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Local Sharp Maximal Functions*

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INTRODUCTION

It is the purpose of this paper to discuss the concept of local sharp maximal functions, to compute the E and K functionals for the couple $\bar{A} = (L^p(R^n), BMO(R^n))$, $0 \leqslant p \leqslant \infty$, and to provide an insight into the theory of various kinds of integral operators, such as Calderón–Zygmund singular integrals, by means of pointwise inequalities involving such sharp maximal functions. The proofs we present here are conceptually different from those in the literature for they rely on the maximal function $M^{\#}_{0,\alpha}$ introduced by John [21] and rediscovered by Strömberg [37].

A word about the five chapters that comprise the paper. Chapter 1 contains the preliminary material, including the definitions of the space $L^0(R^n)$ and of the approximation functional E. In Chapter 2 we prove the "basic inequality," i.e., roughly speaking the principle which allows us to control the oscillation of a function f in a cube Q by means of the (local) sharp maximal function $M^{\#}_{0,\alpha;Q} f$. The quantitative formulation of this principle is given by an estimate in the spirit of the original statement of the John–Nirenberg inequality, namely,

$$|\{x \in Q\colon |f(x) - m_f(Q)| > t,\ M^{\#}_{0,\alpha;Q} f(x) \leqslant \beta t\}|$$
$$\leqslant c_1 e^{-c_2/\beta} \|f - m_f(Q)\|^p_{L^{p,q}(Q)}/t^p,$$

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where $m_f(Q)$ is a median value of f over Q, c_1, c_2 are absolute constants and $0 \leqslant p < \infty$, $0 < q \leqslant \infty$. Chapter 3 contains the explicit computation of the K (and also the E when $p = 0$) functional for the couple $\bar{A} = (L^{p,q}(R^n), BMO(R^n))$, $0 < p < \infty$, $0 < q \leqslant \infty$. In particular we show that

$$K(t, f; L^{1,\infty}(R^n), BMO(R^n)) \approx \sup_{0 < s < t} s(M^{\#}_{0,\alpha} f)^*(s). \tag{1}$$

There is an underlying principle implicit in this statement, to wit, a known expression involving $L^\infty(R^n)$ will remain valid when we put $BMO(R^n)$ in its place provided we replace f^* by $(M^{\#}_{0,\alpha} f)^*$. As for (1) we have in mind the expression

$$K(t, f; L^{1,\infty}(R^n), L^\infty(R^n)) \approx \sup_{0 < s < t} sf^*(s).$$

In Chapter 4 we show that the subadditive operators which map $L^\infty(R^n)$ into $BMO(R^n)$ and $L^1(R^n)$ into $L^{1,\infty}(R^n)$ continuously, are precisely those mappings T for which

$$|\{x \in R^n \colon M^{\#}_{0,\alpha} \operatorname{Tf}(x) > \lambda\}| \leqslant c\, |\{x \in R^n \colon \operatorname{Mf}(x) > c_1 \lambda\}|, \qquad \lambda > 0,$$

where Mf denotes the Hardy–Littlewood maximal function of f. The control in probability, actually an E-functional inequality expressed by this estimate, can be improved to the pointwise inequality

$$M^{\#}_{0,\alpha} \operatorname{Tf}(x) \leqslant c \operatorname{Mf}(x)$$

for a wide class of operator, including some pseudodifferential operators of order zero. Coifman and Meyer [8] have observed that estimates such as $M^{\#} \operatorname{Tf}(x) \leqslant c \operatorname{Mf}(x)$ cannot hold when T = Hilbert transform on R^1. These estimates are important because they imply the continuity of the operators in weighted L^p spaces as well as provide vector-valued inequalities. We conclude by restating in Chapter 5 the results of Garnett and Jones [13] concerning the distance in $BMO(R^n)$ to $L^\infty(R^n)$ to show that for the pair $\bar{A} = (L^\infty(R^n), BMO(R^n))$ we have

$$K(t, f; \bar{A}) \approx \|\bar{M}^{\#}_{0,e^{-t}} f\|_\infty, \qquad t > 0.$$

where $\bar{M}^{\#}_{0,\alpha}$ is a variant of $M^{\#}_{0,\alpha}$.

1. THE K AND E FUNCTIONALS

We begin by recalling the definitions of the spaces we consider. The Lebesgue spaces $L^p(R^n)$, $0<p<\infty$, are as usual the (equivalence class of) complex-valued, Lebesgue measurable functions f on R^n such that

$$\|f\|_p=\left(\int_{R^n}|f(x)|^p\,dx\right)^{1/p}<\infty.$$

In the limiting cases $p=\infty$ and $p=0$ we require that

$$\|f\|_\infty=\text{ess sup}\,|f(x)|<\infty$$

and

$$\|f\|_0=\int_{\{f\neq 0\}}dx<\infty,$$

respectively.

The Lorentz spaces $L^{p,q}(R^n)$, $0<p\leqslant\infty$, $0<q\leqslant\infty$, are those measurable functions f on R^n such that

$$\|f\|_{p,q}=\left(\frac{q}{p}\int_0^\infty (t^{1/p}f^*(t))^q\,\frac{dt}{t}\right)^{1/q}<\infty, \qquad 0<p,\ q<\infty,$$

$$\|f\|_{p,\infty}=\sup t^{1/p}f^*(t)<\infty, \qquad 0<p<\infty$$

and

$$\|f\|_{\infty,\infty}=\|f\|_\infty<\infty,$$

respectively. As is customary we have denoted with $f^*(t)$ the nonincreasing rearrangement of f. More precisely, if we let $|\mathscr{E}|$ denote the Lebesgue measure of a measurable set $\mathscr{E}$ in R^n, and we set

$$m(f,\lambda)=|\{x\in R^n\colon |f(x)|>\lambda\}|,$$

then

$$f^*(t)=\inf\{\lambda\geqslant 0\colon m(f,\lambda)\leqslant t\},$$

where $\inf\varnothing=\infty$. At least formally $\|f\|_{p,q}\to\|f\|_\infty$ as p tends to ∞ and $\|f\|_{p,q}^p\to\|f\|_0$ as p tends to 0 for each fixed q with $0<q\leqslant\infty$. These observations motivate the convenient identifications $\|f\|_{\infty,q}=\|f\|_\infty$ and $\|f\|_{0,q}=\|f\|_0$ for $0<q\leqslant\infty$.

If $\mathscr{E}$ is a Lebesgue measurable set in R^n we shall use the notation $L^{p,q}(\mathscr{E})$ to denote those functions f whose restrictions to $\mathscr{E}$, $f\chi_{\mathscr{E}}$, belong to $L^{p,q}(R^n)$ and we put $\|f\|_{p,q;\mathscr{E}}=\|f\chi_{\mathscr{E}}\|_{p,q}$.

For $1 \leqslant q \leqslant p < \infty$, $\|\cdot\|_{p,q}$ is a norm under which $L^{p,q}(R^n)$ forms a Banach space; for $1 < p < q \leqslant \infty$ the same quantity is equivalent to a norm under which the corresponding Lorentz class forms a Banach space. In the other cases we only obtain metric F spaces: for further details see [28].

By a change of variables we see that

$$\|f\|_{p,q}^q = \frac{q^2}{p} \int_0^\infty m(f, \lambda)^{q/p} \lambda^{q-1}\, d\lambda, \qquad 0 < p,\ q < \infty, \tag{1.1}$$

and we recall that if $1 \leqslant q < p$ or $1 < p < q \leqslant \infty$, then

$$\|f\|_{p,q} \approx \sup \left| \int_{R^n} f(x)\, g(x)\, dx \right|, \tag{1.2}$$

where the sup is taken over all measurable functions g such that

$$\|g\|_{p',q'} \leqslant 1 \qquad \text{with} \quad 1/p + 1/p' = 1/q + 1/q' = 1.$$

Finally $BMO(R^n)$ is the John–Nirenberg space of (equivalence classes modulo constants of) complex-valued, locally summable functions f on R^n such that

$$\|f\|_* = \sup_Q \inf_c \frac{1}{|Q|} \int_Q |f(x) - c|\, dx < \infty,$$

where the inf is taken over all complex numbers c and the sup over all finite cubes Q in R^n with sides parallel to the coordinate axes. An equivalent formulation is as follows: set

$$M^\# f(x) = \sup_{x \in Q} \inf_c \frac{1}{|Q|} \int_Q |f(y) - c|\, dy$$

and let $BMO(R^n) = \{f\colon \|f\|_* \equiv \|M^\# f\|_\infty < \infty\}$.

The spaces described above are all examples of normed Abelian groups. For a couple $\bar{A} = (A_0, A_1)$ of these, with A_0 and A_1 continuously embedded in a Hausdorff topological vector space, Peetre's K-functional is defined by

$$K(t, a; \bar{A}) = \inf_{a = a_0 + a_1} (\|a_0\|_{A_0} + t\, \|a_1\|_{A_1})$$

for $t > 0$ and $a \in A_0 + A_1$. The intermediate interpolation spaces $\bar{A}_{\theta,r}$ are defined for $0 < \theta < 1$ as those functions $a \in A_0 + A_1$ such that

$$\|a\|_{\bar{A}_{\theta,r}} = \left(\int_0^\infty (t^{-\theta} K(t, a; \bar{A}))^r \frac{dt}{t} \right)^{1/r} < \infty, \qquad 0 < r < \infty$$

and

$$\|a\|_{\bar{A}_{\theta,\infty}} = \sup t^{-\theta} K(t, a; \bar{A}) < \infty,$$

respectively. A closely related concept to that of K-functional is the best approximation E defined by

$$E(t, a; \bar{A}) = \inf_{\|a_1\|_{A_1} \leqslant t} \|a - a_1\|_{A_0}$$

for $t > 0$ and $a \in A_0 + A_1$.

As a complement to earlier results [29] it was observed in [20] that the connection between the K and E functionals can be expressed as follows:

LEMMA 1.1. *As a function of t, $K(t, a; \bar{A})/t$ is equivalent to the right-continuous inverse of $E(t, a; \bar{A})/t$; more precisely*

$$K(t, a; \bar{A})/2t \leqslant (E(t, a; \bar{A})/t)^{-1} \leqslant K(t, a; \bar{A})/t.$$

Proof. Put $K_\infty(t, a; \bar{A}) = \inf_{a = a_0 + a_1} \max(\|a_0\|_{A_0}, t \|a_1\|_{A_1})$. Clearly $K_\infty(t, a; \bar{A}) \leqslant K(t, a; \bar{A}) \leqslant 2K_\infty(t, a; \bar{A})$. A look at the Gagliardo diagram $\Gamma = \Gamma(a, \bar{A}) = \{(x_0, x_1) \in R^2 \colon a = a_0' + a_1$ with $\|a_0\|_{A_0} \leqslant x_0$ and $\|a_1\|_{A_1} \leqslant x_1\}$, see Fig. 1.1, reveals that $K_\infty(t, a; \bar{A})/t$ is the right-continuous inverse of $E(t, a; \bar{A})/t$ and this proves the lemma.

We also introduce the approximation spaces $\bar{A}_{p,q;E}$ as those functions $a \in A_0 + A_1$ such that

$$\|a\|_{\bar{A}_{p,q;E}} = \left(\int_0^\infty (tE(t, a; \bar{A})^{1/p})^q \frac{dt}{t} \right)^{1/q} < \infty, \qquad 0 < p,\ q < \infty$$

and

$$\|a\|_{\bar{A}_{p,\infty;E}} = \sup tE(t, a; \bar{A})^{1/p} < \infty, \qquad 0 < p < \infty,$$

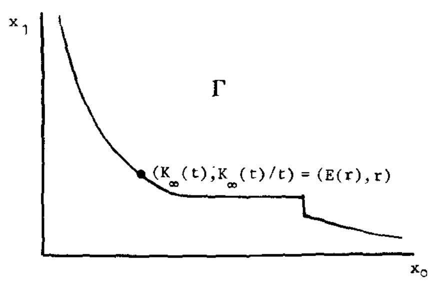

FIGURE 1.1

respectively. From Lemma 1.1 and a change of variables we have that $(\bar{A}_{\theta,r})^{1/\theta} \approx \bar{A}_{p,q;E}$, i.e.,

$$\|a\|_{\bar{A}_{\theta,r}}^{1/\theta} \approx \|a\|_{\bar{A}_{p,q;E}}, \tag{1.3}$$

where $p = \theta/(1-\theta)$ and $q = \theta r$.

2. Local Maximal Functions

Of the many ways to measure the oscillation of a function the one we study here in some detail is that provided by a maximal operator introduced by John [21] and Strömberg [37]. As we shall see it has many advantages, one being that it is a priori defined for arbitrary measurable functions rather than locally summable, say. So let f be a complex-valued, Lebesgue measurable function on R^n. For $0 < \alpha \leqslant \frac{1}{2}$ we put

$$M_{0,\alpha}^{\#} f(x) = \sup_{x \in Q} \inf_{c} \inf\{A \geqslant 0 \colon |\{y \in Q \colon |f(y) - c| > A\}| < \alpha\, |Q|\}, \tag{2.1}$$

where c runs over all complex numbers and Q is an arbitrary finite cube in R^n with sides parallel to the coordinate axes. This restriction on the cubes Q is assumed throughout the rest of the paper and is therefore not explicitly stated any more. Often it is not important to take the inf over c since there are optimal choices, namely the median values. A median value $m_f(Q)$ of a real-valued function f over the cube Q is a, possibly nonunique, real number such that

$$|\{x \in Q \colon f(x) > m_f(Q)\}| \leqslant |Q|/2,$$
$$|\{x \in Q \colon f(x) < m_f(Q)\}| \leqslant |Q|/2.$$

For $f = f_1 + i f_2$ complex-valued, we set $m_f(Q) = m_{f_1}(Q) + i m_{f_2}(Q)$.

It is readily seen that

$$|\{y \in Q \colon |f(y) - m_f(Q)| > 10 M_{0,\alpha}^{\#} f(x)\}| < \alpha\, |Q| \tag{2.2}$$

for an arbitrary $x \in Q$, and this justifies the above assertion.

From the definition we immediately get that $M_{0,\alpha}^{\#} f(x)$ is a lower semicontinuous function of x and that

$$M_{0,\alpha}^{\#}(f+g)(x) \leqslant 2(M_{0,\alpha/2}^{\#} f(x) + M_{0,\alpha/2}^{\#}\, g(x)).$$

Less trivial properties follow by comparison with another maximal operator, $M_{0,\alpha}f(x)$, defined for complex-valued, Lebesgue measurable functions f on R^n and $0<\alpha\leqslant\frac{1}{2}$, by

$$M_{0,\alpha}f(x)=\sup_{x\in Q}\inf\{A\geqslant 0\colon |\{y\in Q\colon |f(y)|>A\}|<\alpha\,|Q|\}. \qquad (2.3)$$

$M_{0,\alpha}f$ is also lower semi-continuous,

$$M_{0,\alpha}(f+g)(x)\leqslant 2(M_{0,\alpha/2}f(x)+M_{0,\alpha/2}\,g(x))$$

and

$$\begin{aligned}|\{y\in R^n\colon |f(y)|>\lambda\}|&\leqslant|\{y\in R^n\colon M_{0,\alpha}f(y)>\lambda\}|\\ &\leqslant 5^n\,|\{y\in R^n\colon |f(y)|>\lambda\}|/\alpha. \qquad (2.4)\end{aligned}$$

Here the left-hand side inequality is an easy consequence of the identity $\{y\in R^n\colon M_{0,\alpha}f(y)>\lambda\}=\{y\in R^n\colon M(\chi_{|f|>t},y)>\alpha\}$, where M denotes the Hardy–Littlewood maximal operator and the right-hand side inequality follows by a usual covering argument.

If Ω is an arbitrary open set in R^n it is possible to define localized versions $M^{\#}_{0,\alpha;\Omega}$ and $M_{0,\alpha;\Omega}$ of the maximal functions $M^{\#}_{0,\alpha}$ and $M_{0,\alpha}$, respectively, by restricting the cubes in (2.1) and (2.3) to subcubes of Ω.

We are now ready to present the "basic inequality" which relates the size of f to that of $M^{\#}_{0,\alpha}f$.

THEOREM 2.1. *Let Q_0 be a fixed cube. If f is a real-valued, Lebesgue measurable function defined on Q_0, there is a constant $c=c(n)$ such that*

$$\begin{aligned}&|\{y\in Q_0\colon |f(y)-m_f(Q_0)|>t,\ M^{\#}_{0,\alpha;Q_0}f(y)\leqslant\beta t/10\}|\\ &\quad\leqslant c(n)\,\alpha\,|\{y\in Q_0\colon |f(y)-m_f(Q_0)|>(1-\beta)\,t,\ M^{\#}_{0,\alpha;Q_0}f(y)\leqslant\beta t/10\}|\end{aligned}$$

for $0<\alpha\leqslant\frac{1}{2}$ and all $0<\beta\leqslant 1$ and $t>0$.

Proof. The proof makes use of the following version of a Whitney-like decomposition, namely

LEMMA 2.2. *Let Q_0 be a fixed cube and suppose that $\mathcal{O}$ is an open set relative to Q_0 which is strictly contained in Q_0. Then there is a sequence $\{Q_k\}_{k=1}^{\infty}$ of cubes such that*

$$\mathcal{O}=\bigcup_k Q_k,$$

$$|Q_i\cap Q_j|=0 \qquad \text{if}\quad i\neq j,\ i,j\geqslant 1,$$

$$\operatorname{diam} Q_k\leqslant\operatorname{dist}(Q_k, Q_0\cap\mathcal{O}^c)\leqslant 4\operatorname{diam} Q_k, \qquad k\geqslant 1.$$

The proof of the lemma only requires simple modifications of the one in [35, p. 167], and is therefore omitted. In fact it follows that the cubes Q_k can be chosen so that diam $Q_k = 2^{-m}$ diam Q_0, for some integer $m = m(k)$, $k \geqslant 1$.

We return now to the proof of the theorem. With no loss of generality we may assume that $m_f(Q_0) = 0$. Let $\mathcal{O}_1$ be an open set in Q_0 which contains

$$\mathcal{U}_1 = \{ y \in Q \colon M^{\#}_{0,\alpha;Q_0} f(y) \leqslant \beta t/10 \}$$

and let $\mathcal{O}$ be another open set containing

$$\begin{aligned}\mathcal{U} = \{ y \in Q_0 \colon |f(y)| > (1-\beta)\, t,\ M^{\#}_{0,\alpha;Q_0} f(y) \leqslant \beta t/10 \}\\ \cup \{ y \in Q_0 \colon M_{0,1/4;\mathcal{O}_1} f(y) > (1-\beta)\, t \}.\end{aligned}$$

Assume first that we can find such a set $\mathcal{O}$ strictly contained in Q_0 and let $\{Q_k\}_{k=1}^{\infty}$ be the decomposition given by Lemma 2.2. The particular nature of this decomposition implies that each Q_k is contained in a cube $Q'_k \subset Q_0$ such that

$$x_k \in Q'_k \cap \mathcal{O}^c \neq \varnothing \qquad \text{and} \qquad \operatorname{diam} Q'_k \leqslant 10 n^{1/2} \operatorname{diam} Q_k, \quad k \geqslant 1. \tag{2.5}$$

Clearly

$$f(x) = \sum_{k \geqslant 1} (f(x) - c_k)\, \chi_{Q_k}(x) + \sum_{k \geqslant 1} c_k \chi_{Q_k}(x) + f(x)\, \chi_{\mathcal{O}^c}(x),$$

where $c_k = m_f(Q'_k)$. Thus

$$\begin{aligned}&|\{ y \in Q_0 \colon |f(y)| > t,\ M^{\#}_{0,\alpha;Q_0} f(y) \leqslant \beta t/10 \}|\\ &\quad\leqslant \sum_{k \geqslant 1} |\{ y \in Q_k \colon |f(y) - c_k| > \beta t,\ M^{\#}_{0,\alpha;Q_0} f(y) \leqslant \beta t/10 \}|\\ &\qquad + \sum_{k \geqslant 1} |\{ y \in Q_k \colon |c_k| > (1-\beta)\, t \}| \equiv \sum_{k=1}^{\infty} A_k + \sum_{k=1}^{\infty} B_k, \qquad \text{say.}\end{aligned}$$

Each B_k in the second sum vanishes. Indeed, for any $Q \subset \mathcal{O}_1$ we have, since f is real-valued, that

$$|m_f(Q)| \leqslant \inf_{x \in Q} M_{0,\alpha';\mathcal{O}_1} f(x), \qquad 0 < \alpha' \leqslant \tfrac{1}{2}.$$

Hence

$$|c_k| \leqslant M_{0,1/4;\mathcal{O}_1} f(x_k) \leqslant (1-\beta)\, t.$$

As for the first sum, from the choice of $\mathcal{O}$ and (2.5) we see that

$$\sum_{k\geqslant 1} A_k \leqslant \sum_{k\geqslant 1} |\{y\in Q'_k: |f(y)-c_k|>\beta t, M^{\#}_{0,\alpha;Q_0}f(y)\leqslant \beta t/10\}|$$
$$\leqslant \alpha \sum_{k=1}^{\infty} |Q'_k| \leqslant \alpha(10n^{1/2})^n\,|\mathcal{O}|.$$

Since $\mathcal{O}\supset\mathcal{U}$ is arbitrary we conclude that

$$\begin{aligned}
&|\{y\in Q_0: |f(y)|>t, M^{\#}_{0,\alpha;Q_0}f(y)\leqslant \beta t/10\}|\\
&\quad\leqslant \alpha(10n^{1/2})^n(|\{y\in Q_0: |f(y)|>(1-\beta)\,t, M^{\#}_{0,\alpha;Q_0}f(y)\leqslant \beta t/10\}|\\
&\qquad + |\{y\in Q_0: M_{0,1/4;\mathcal{O}_1}f(y)>(1-\beta)\,t\}|)\\
&\quad\leqslant \alpha(50n^{1/2})^n\,|\{y\in Q_0: |f(y)|>(1-\beta)\,t, M^{\#}_{0,\alpha;Q_0}f(y)\leqslant \beta t/10\}|,
\end{aligned}$$

according to (2.4) since $\mathcal{O}_1$ is arbitrarily close, in measure, to $\mathcal{U}_1$. This completes the proof, with $c(n)=(50n^{1/2})^n$, provided we can find $\mathcal{O}$ strictly contained in Q_0. However, if this is not possible it must be because one of the sets

$$\{y\in Q_0: |f(y)|>(1-\beta)\,t, M^{\#}_{0,\alpha;Q_0}f(y)\leqslant \beta t/10\}$$

or

$$\{y\in Q_0: M_{0,1/4;\mathcal{O}_1}f(y)>(1-\beta)\,t\}$$

is essentially Q_0. In any case, from (2.4) it follows that

$$20^{-n}\,|Q_0|\leqslant |\{y\in \mathcal{O}_1: |f(y)|>(1-\beta)\,t\}|.$$

Since $m_f(Q_0)=0$, by (2.2) we have that

$$\begin{aligned}
&|\{y\in Q_0: |f(y)|>t, M^{\#}_{0,\alpha;Q_0}f(y)\leqslant \beta t/10\}|\\
&\quad\leqslant |\{y\in Q_0: |f(y)|>10 \inf_{x\in Q_0} M^{\#}_{0,\alpha;Q_0}f(x)\}|\leqslant \alpha\,|Q_0|\\
&\quad\leqslant \alpha 20^n\,|\{y\in \mathcal{O}_1: |f(y)|>(1-\beta)\,t\}|.
\end{aligned}$$

As the measure of the set $\{y\in \mathcal{O}_1: |f(y)|>(1-\beta)\,t\}$ can again be assumed to be arbitrarily close to that of $\{y\in Q_0: |f(y)|>(1-\beta)\,t, M^{\#}_{0,\alpha;Q_0}f(y)\leqslant \beta t\}$ our proof is complete.

A theorem in the spirit of the above result with $M^{\#}f$ in place of $M^{\#}_{0,\alpha;Q_0}f$ goes back to Fefferman and Stein [12].

The basic inequality has a number of interesting consequences. We mention here only those which play a role in what follows. First we characterize the spaces obtained by applying $L^{p,q}$ norms to $M^{\#}_{0,\alpha}$; starting with the case $p=0$ we have

COROLLARY 2.3. *There is an* $\alpha_0=\alpha_0(n)$ *such that for* $0<\alpha\leqslant\alpha_0(n)$ *there exist constants* $c_j=c_j(\alpha,n)$, $j=1,2$, *so that for each measurable function* f *there is a constant* c_f *such that*

$$c_1\,\|f-c_f\|_0\leqslant\|M^{\#}_{0,\alpha}f\|_0\leqslant c_2\,\|f-c_f\|_0.$$

Proof. The right-hand side inequality is clear. Indeed, since for any constant c we have $M^{\#}_{0,\alpha}f(y)\leqslant M_{0,\alpha}(f-c)(y)$, from (2.4), it follows that

$$|\{y\in R^n\colon M^{\#}_{0,\alpha}f(y)>t\}|\leqslant 5^n\,|\{y\in R^n\colon|f(y)-c|>t\}|/\alpha.$$

Thus letting t decrease to 0 we get the desired inequality with $c_2=5^n/\alpha$. This inequality is true for each c, but of course it is trivial except for a unique value.

As for the left-hand side inequality it follows from

LEMMA 2.4. *Let* $0<\alpha\leqslant\frac{1}{4}$. *If* $Q_0\subseteq Q_1$ *are two cubes with* $|Q_1|\leqslant 2\,|Q_0|$, *then*

$$|m_f(Q_1)-m_f(Q_0)|\leqslant 20\inf_{x\in Q_0}M^{\#}_{0,\alpha}f(x).$$

Proof. Suppose the desired conclusion does not hold. Then $\{y\in Q_0\colon|f(y)-m_f(Q_0)|\leqslant 10\inf_{x\in Q_0}M^{\#}_{0,\alpha}f(x)\}$ and $\{y\in Q_0\colon|f(y)-m_f(Q_1)|\leqslant 10\inf_{x\in Q_0}M^{\#}_{0,\alpha}f(x)\}$ are disjoint sets and by (2.2) it follows that

$$\begin{aligned}2\alpha\,|Q_0|\geqslant\alpha\,|Q_1|&\geqslant|\{y\in Q_1\colon|f(y)-m_f(Q_1)|>10\inf_{x\in Q_0}M^{\#}_{0,\alpha}f(x)\}|\\&\geqslant|\{y\in Q_0\colon|f(y)-m_f(Q_0)|\leqslant 10\inf_{x\in Q_0}M^{\#}_{0,\alpha}f(x)\}|\geqslant(1-\alpha)\,|Q_0|,\end{aligned}$$

which is a contradiction.

We return to the proof of the corollary. If $M^{\#}_{0,\alpha}f\in L^0(R^n)$, then for Q_0 sufficiently large we have that $\inf_{x\in Q_0}M^{\#}_{0,\alpha}f(x)=0$. Let $\{Q_k\}_{k\geqslant 0}$ be an increasing sequence of cubes with $\bigcup_{k\geqslant 0}Q_k=R^n$. By Lemma 2.4 we know that $m_f(Q_0)=m_f(Q_k)$, $k\geqslant 1$. Also the basic inequality applied to each Q_k with $0<\alpha\leqslant 1/2c(n)$ gives that

$$\begin{aligned}&|\{y\in Q_k\colon|f(y)-m_f(Q_0)|>t\}|\\&\quad\leqslant|\{y\in Q_k\colon|f(y)-m_f(Q_0)|>t,\ M^{\#}_{0,\alpha;Q_k}f(y)\leqslant\beta t/10|\}\\&\qquad+|\{y\in Q_k\colon M^{\#}_{0,\alpha;Q_k}f(y)>\beta t/10\}|\\&\quad\leqslant\tfrac{1}{2}(\|f-m_f(Q_0)\|_{0;Q_k}+\|M^{\#}_{0,\alpha}f\|_0),\end{aligned}$$

whence the conclusion follows by first letting t tend to 0 so that

$$\| f - m_f(Q_0) \|_{0;Q_k} \leqslant \| M^{\#}_{0,\alpha} f \|_0$$

and then letting k tend to infinity.

In Corollary 2.5 we discuss the other end-point case, namely $p = \infty$. We have the following result due to John and Strömberg.

COROLLARY 2.5. *There is an* $\alpha_0 = \alpha_0(n)$ *such that for* $0 < \alpha \leqslant \alpha_0(n)$, *there exist constants* $c_j = c_j(\alpha, n)$, $j = 1, 2$, *so that for each measurable function* f *we have*

$$c_1 \| f \|_* \leqslant \| M^{\#}_{0,\alpha} f \|_\infty \leqslant c_2 \| f \|_*.$$

Proof. For each fixed cube Q and each $\varepsilon > 0$ there is a constant c' such that

$$\frac{1}{|Q|} \int_Q |f(y) - c'| \, dy \leqslant (1 + \varepsilon) \inf_c \frac{1}{|Q|} \int_Q |f(y) - c| \, dy.$$

Hence by Chebyshev's inequality we see that

$$|\{ y \in Q \colon |f(y) - c'| > t \}| \leqslant \int_Q |f(y) - c'| \, dy/t \leqslant (1 + \varepsilon) \, |Q| \, \| f \|_*/t.$$

Thus we readily see that

$$\| M^{\#}_{0,\alpha} f \|_\infty \leqslant \| f \|_*/\alpha.$$

On the other hand for any large N and a fixed cube Q

$$\begin{aligned} I(N) &= \int_0^N |\{ y \in Q \colon |f(y) - m_f(Q)| > t \}| \, dt \\ &\leqslant 10 \, \| M^{\#}_{0,\alpha} f \|_\infty \, |Q| + \int_{10 \| M^{\#}_{0,\alpha} f \|_\infty}^N |\{ y \in Q \colon |f(y) - m_f(Q)| > t \}| \, dt. \end{aligned}$$

By taking $\beta t/10 = \| M^{\#} f \|_\infty$ and $\alpha \leqslant \alpha_0(n) = 1/4c(n)$ in the basic inequality we see that the last integral can be estimated by $I(N)/2$. Hence

$$I(N)/2 \leqslant 20 \, \| M^{\#}_{0,\alpha} f \|_\infty \, |Q|,$$

and by letting N tend to infinity, taking inf over all c and sup over Q we have

$$\| f \|_* \leqslant \sup_Q \frac{1}{|Q|} \int_Q |f(y) - m_f(Q)| \, dy \leqslant 40 \, \| M^{\#}_{0,\alpha} f \|_\infty,$$

as we wanted to show.

Finally for the intermediate values of p we have

COROLLARY 2.6. *Let $0<p<\infty$, $0<q\leqslant\infty$. There is an $\alpha_0=\alpha_0(n)$ such that for $\alpha\leqslant\alpha_0$ there exist constants $c_j=c_j(\alpha,n)$, $j=1,2$, so that for each measurable function f there is a constant c_f such that*

$$c_1\,\|f-c_f\|_{p,q}\leqslant\|M^{\#}_{0,\alpha}f\|_{p,q}\leqslant c_2\,\|f-c_f\|_{p,q}.$$

Proof. The right-hand side inequality is obvious and follows as in the proof of Corollary 2.3 now with $c_2=(5^n/\alpha)^{1/p}$. As for the left-hand side inequality, to fix ideas we prove the case $p=q$, the other cases requiring only trivial modifications using (1.1). Let βQ denote the cube with the same center as Q and with side length β times that of Q. If $M^{\#}_{0,\alpha}f$ is in $L^p(R^n)$, then

$$\inf_{x\in Q} M^{\#}_{0,\alpha}f(x)\leqslant\|M^{\#}_{0,\alpha}f\|_p\,|Q|^{-1/p}.$$

Hence by Lemma 2.4 we see that if Q_0 denotes the unit cube centered at the origin, then $m_f(2^kQ_0)$ converges to a number c_f as k tends to infinity and that

$$|m_f(2^kQ_0)-c_f|\leqslant 20(1-2^{-n/p})^{-1}\,2^{-kn/p}\,\|M^{\#}_{0,\alpha}f\|_p.$$

Set $A_k=20(1-2^{-n/p})^{-1}\,2^{-kn/p}\,\|M^{\#}_{0,\alpha}f\|_p$. Clearly for each large N

$$\begin{aligned}
&p\int_0^N|\{y\in 2^kQ_0\colon |f(y)-c_f|>t\}|\,t^{p-1}\,dt\\
&\quad\leqslant p\int_0^N|\{y\in 2^kQ_0\colon |f(y)-m_f(2^kQ_0)|>t/2\}|\,t^{p-1}\,dt\\
&\qquad+p\int_0^{2A_k}|\{y\in 2^kQ_0\colon |m_f(2^kQ_0)-c_f|>t/2\}|\,t^{p-1}\,dt.
\end{aligned}$$

The second summand in the above sum is less than $(2A_k)^p\,|2^kQ_0|=c_p^p\,\|M^{\#}_{0,\alpha}f\|_p^p$ with $c_p=40(1-2^{-n/p})^{-1}$. As for the first, we can use the trivial estimate

$$\begin{aligned}
&|\{y\in 2^kQ_0\colon |f(y)-m_f(2^kQ_0)|>t/2\}|\\
&\quad\leqslant|\{y\in 2^kQ_0\colon |f(y)-m_f(2^kQ_0)|>t/2,\ M^{\#}_{0,\alpha}f(y)\leqslant t/40\}|\\
&\qquad+|\{y\in 2^kQ_0\colon M^{\#}_{0,\alpha}f(y)>t/40\}|,
\end{aligned}$$

and the basic inequality with $\beta=\frac{1}{2}$ and $\alpha\leqslant\alpha_0=1/c(n)\,4^{p+1}$. We then see that

$$\tfrac{3}{4}p\int_0^N|\{y\in 2^kQ_0\colon |f(y)-c_f|>t\}|\,t^{p-1}\,dt\leqslant c_p'\,\|M^{\#}_{0,\alpha}f\|_p^p$$

with $c'_p = 40^p + c_p^p$ and we obtain the desired conclusion by first letting N and then k tend to infinity.

Using the ideas in the proof of the basic inequality, it is possible to prove an extension of John–Nirenberg's lemma.

THEOREM 2.7. *Let $0 \leqslant p < \infty$, $0 < q \leqslant \infty$ and let Q_0 be a fixed cube. Then there are constants $c_1 = c_1(\alpha, n)$ and $c_2 = c_2(\alpha, n)$ such that*

$$|\{y \in Q_0 \colon |f(y) - m_f(Q_0)| > t,\ M^{\#}_{0,\alpha;Q_0} f(y) \leqslant \beta t\}| \\ \leqslant c_1 e^{-c_2/\beta} \|f - m_f(Q_0)\|^p_{p,q;Q_0}/t^p,$$

provided that $\alpha \leqslant \alpha_0(n)$ is sufficiently small.

Proof. There is no loss of generality in assuming that f is real-valued. In case $\beta \geqslant \frac{1}{100}$, Chebyshev's inequality yields the required estimate. So assume that $1/(k+1) < \beta \leqslant 1/k$ for some integer $k \geqslant 100$. The basic inequality in Theorem 2.1 states that

$$I = |\{y \in Q_0 \colon |f(y) - m_f(Q_0)| > t,\ M^{\#}_{0,\alpha;Q_0} f(y) \leqslant \beta t/10\}| \\ \leqslant c(n)\,\alpha\, |\{y \in Q_0 \colon |f(y) - m_f(Q_0)| > (1-\beta)\,t,\ M^{\#}_{0,\alpha;Q_0} f(y) \leqslant \beta t/10\}|.$$

Put $\bar{t} = (1-\beta)\,t$, $\bar{\beta} = \beta/(1-\beta)$ and note that the right-hand side can then be estimated by

$$c(n)\,\alpha\, |\{y \in Q_0 \colon |f(y) - m_f(Q_0)| > (1-\bar{\beta})\,\bar{t}\ M^{\#}_{0,\alpha;Q_0} f(y) \leqslant \bar{\beta}\bar{t}/10\}|.$$

Since $(1-\bar{\beta})\,\bar{t} = (1-2\beta)\,t$ and $\bar{\beta}\bar{t} = \beta t$ we readily see that

$$I \leqslant (c(n)\,\alpha)^2\, |\{y \in Q_0 \colon |f(y) - m_f(Q_0)| > (1-2\beta)\,t,\ M^{\#}_{0,\alpha;Q_0} f(y) \leqslant \beta t/10\}|.$$

Since $1 - k\beta \geqslant 0$ we may iterate this procedure j times for any integer j so that $1 < j \leqslant k$ obtaining

$$I \leqslant (c(n)\,\alpha)^j\, |\{y \in Q_0 \colon |f(y) - m_f(Q_0)| > (1-j\beta)\,t,\ M^{\#}_{0,\alpha;Q_0} f(y) \leqslant \beta t/10\}|.$$

By Chebyshev's inequality the right-hand side above can be estimated by

$$(c(n)\,\alpha)^j\, \|f - m_f(Q_0)\|^p_{p,q;Q_0}/t^p(1-j\beta)^p.$$

Thus choosing $j = 1/[\beta/2]$, α appropriately small and making the trivial change from $\beta/10$ to β in I, proves the theorem.

COROLLARY 2.8 (John–Nirenberg lemma). *Let Q be a fixed cube and let*

$A_f(Q)=\sup_{x\in Q} M^{\#}_{0,\alpha;Q}f(x)$. *Then there is an* $\alpha_0=\alpha_0(n)$ *such that for* $\alpha\leqslant\alpha_0$ *there exist constants* $c_1=c_1(\alpha,n)$, $c_2=c_2(\alpha,n)$ *so that*

$$|\{y\in Q\colon |f(y)-m_f(Q)|>t\}|\leqslant c_1 e^{-c_2t/A_f(Q)}\,|Q|.$$

Proof. If $A_f(Q)=+\infty$, there is nothing to prove, so we may assume that $A_f(Q)$ is finite. By considering the real and imaginary parts of f separately, we may assume that f is real-valued. Again if t is small, $t\leqslant A_f(Q)$ say, there is nothing to show, so we may assume that $t>A_f(Q)$. In Theorem 2.7, now set $\beta=A_f(Q)/t$, $p=0$ and $0<q$ anything, and observe that $M^{\#}_{0,\alpha;Q}f(y)\leqslant\beta t=A_f(Q)$ for each $y\in Q$ and that $\|(f-m_f(Q))\,\chi_Q\|_0\leqslant |Q|$. Therefore

$$\begin{aligned}|\{y\in Q\colon|f(y)-m_f(Q)>t\}|&=|\{y\in Q\colon|f(y)-m_f(Q)|>t, M^{\#}_{0,\alpha,Q}f(y)\leqslant\beta t\}|\\&\leqslant c_1 e^{-c_2t/A_f(Q)}\,|Q|\end{aligned}$$

provided that α is sufficiently small. This is the desired result.

Remark 2.9. The estimate

$$\begin{aligned}|\{y\in Q_0\colon |f(y)-m_f(Q_0)|>t\}|&\leqslant c(n)\,\alpha(|\{y\in Q_0\colon M^{\#}_{0,\alpha;Q_0}f(y)>\beta t/10\}|\\&\quad+|\{y\in Q_0\colon |f(y)-m_f(Q_0)|>(1-\beta)\,t\}|)\end{aligned}\tag{2.6}$$

for $0<\alpha\leqslant\frac{1}{2}$, $0<\beta<1$, $t>0$ and $\beta t\geqslant 10\inf_{x\in Q_0} M^{\#}_{0,\alpha;Q_0}f(x)$, is proved in a similar, yet somewhat simpler way than Theorem 2.1. Nevertheless, the interested reader can verify that (2.6) suffices to yield Corollaries 2.3, 2.5, 2.6, and 2.8.

Remark 2.10. There is also a version of Theorem 2.7 with the finite cube Q_0 replaced by R^n, cf. Corollary 2.2. The quantity $\|f-m_f(Q_0)\|_{p,q;Q_0}$ is then replaced by $\|f-c_f\|_{p,q}$ or equivalently by $\|M^{\#}_{0,\alpha}f\|_{p,q}$ (cf. [38]).

Remark 2.11. We recall first a couple of definitions. We say that a positive, locally summable function $w(x)$ defined on R^n is a doubling weight if there is a number $d\geqslant n$ and a constant c such that for each x in R^n and for each $r>0$ and $t\geqslant 1$ we have that

$$w(B(x,tr))=\int_{\{y\in R^n:|x-y|\leqslant tr\}} w(y)\,dy\leqslant ct^d w(B(x,r)).$$

It is a well-known fact that many of the results which hold true for Lebesgue measure remain valid for doubling weights. Here this is the case under the appropriate circumstances, namely, let

$$M^{\#}_{0,\alpha;w}f(x)=\sup_{x\in Q}\inf_{p}\inf\{A\geqslant 0\colon w(\{y\in Q\colon |f(y)-p(y)|>A\})<\alpha w(Q)\},$$

where p runs over the polynomials of degree $m = m(d)$. Then basically all weighted analogues remain valid. Assume that in addition w is an $A_\infty(R^n)$ weight, that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathscr{E}$ is an arbitrary measurable subset of a cube Q and $|\mathscr{E}|/|Q| \leqslant \delta$, then $w(\mathscr{E})/w(Q) \leqslant \varepsilon$. Then in this case there is an $\alpha' = \alpha'(\alpha, n, w)$ such that

$$M^{\#}_{0,\alpha;w} f(x) \leqslant c M^{\#}_{0,\alpha'} f(x).$$

The consideration of weights also allows us to extend the above results in a different direction, namely, to the consideration of the vector-valued analogues. In this setting we have

THEOREM 2.12. *Let* $0 < p < \infty$, $0 < q \leqslant \infty$, *and* $0 < r < \infty$. *There is an* $\alpha_0 = \alpha_0(n)$ *such that for* $\alpha \leqslant \alpha_0$, *there exist constants* k_1, k_2 *(depending on* α *and n alone) so that to each sequence* $\{f_j\}_{j \geqslant 1}$ *of measurable functions on* R^n *there corresponds a sequence* $\{c_j\}_{j \geqslant 1}$ *of complex numbers such that*

$$k_1 \left\| \left(\sum_j |f_j - c_j|^r \right)^{1/r} \right\|_{p,q} \leqslant \left\| \left(\sum_j (M^{\#}_{0,\alpha} f_j)^r \right)^{1/r} \right\|_{p,q} \leqslant k_2 \left\| \left(\sum_j |f_j - c_j|^r \right)^{1/r} \right\|_{p,q}.$$

Proof. For $s > 0$ set

$$M_s f(x) = \sup_{x \in Q} \left(\frac{1}{|Q|} \int_Q |f(y)|^s \, dy \right)^{1/s}.$$

Clearly for any constant c and $0 < \alpha \leqslant \frac{1}{2}$ we have

$$M^{\#}_{0,\alpha} f(x) \leqslant M_s(f - c)(x)/\alpha^{1/s}.$$

Now let $s < \min(p, r)$. By the vector-valued version of the Hardy–Littlewood maximal theorem due to Fefferman and Stein (11), it follows that

$$\left\| \left(\sum_j (M^{\#}_{0,\alpha} f_j)^r \right)^{1/r} \right\|_{p,q} \leqslant k_2 \left\| \left(\sum_j |f_j - c_j|^r \right)^{1/r} \right\|_{p,q}$$

for arbitrary constants $\{c_j\}_{j \geqslant 1}$. Of course there is only one choice of the c_j's for which the inequality is possibly nontrivial.

To prove the other inequality we use duality. If $I = \|(\sum_j (M^{\#}_{0,\alpha} f_j)^r)^{1/r}\|_{p,q} < \infty$, then by Corollary 2.6 there are unique constants

c_j's such that $f_j - c_j$ is in $L^{p,q}(R^n)$ for each $j \geqslant 1$. This is our choice for the c_j's. Now choose t so that

$$1 < r/t,\ p/t < \infty, \qquad 1 < q/t \leqslant \infty.$$

Then by (1.2)

$$\left\| \left(\sum_j |f_j - c_j|^r \right)^{1/r} \right\|_{p,q}^t \leqslant \sup \int_{R^n} \sum_j |f_j(y) - c_j|^t g_j(y)\, dy,$$

where the sup is taken over all sequences $\{g_j\}_{j \geqslant 1}$ with

$$\left\| \left(\sum_j |g_j|^{(r/t)'} \right)^{1/(r/t)'} \right\|_{(p/t)',(q/t)'} \leqslant 1.$$

Moreover $|g_j(y)| \leqslant M_s g_j(y)$ for each s and $M_s g_j$ satisfies the $A_\infty(R^n)$ condition with the ε's and δ's in the definition independent of j if now $s > 1$ (see [9]). Hence by the known weighted results, see Remark 2.11 and [26], we get

$$\begin{aligned} \int_{R^n} |f_j(y) - c_j|^t g_j(y)\, dy &\leqslant \int_{R^n} |f_j(y) - c_j|^t M_s g_j(y)\, dy \\ &\leqslant c \int_{R^n} (M^{\#}_{0,\alpha} f_j(y))^t M_s g_j(y)\, dy. \end{aligned} \tag{2.7}$$

Again by the results of Fefferman and Stein we have that

$$\begin{aligned} &\left\| \left(\sum_j (M_s g_j)^{(r/t)'} \right)^{1/(r/t)'} \right\|_{(p/t)',(q/t)'} \\ &\qquad \leqslant c \left\| \left(\sum_j |g_j|^{(r/t)'} \right)^{1/(r/t)'} \right\|_{(p/t)',(q/t)'} \leqslant c \end{aligned}$$

if $s < \min((r/t)', (p/t)')$. From this fact and Hölder's inequality, (2.7) yields the desired conclusion at once.

Remark 2.13. Theorem 2.12 also holds for doubling weights if we replace $M^{\#}_{0,\alpha}$ by $M^{\#}_{0,\alpha;w}$ and the constant c_f by a polynomial. This is clear from the above proof once we observe that Fefferman–Stein's vector-valued theorem remains true in the weighted case if the Hardy maximal function M is replaced by its weighted analogue. The appropriate Lorentz spaces $L^{p,q}_w(R^n)$ in this case are composed of those functions f such that

$$\|f\|_{p,q;w} = \left(p \int_0^\infty w(\{y \in R^n : |f(y)| > t\})^{q/p}\, t^{q-1}\, dt \right)^{1/q} < \infty,$$

$$0 < p < \infty,\ 0 < q < \infty$$

and

$$\|f\|_{p,\infty;w} = \sup tw(\{y \in R^n : |f(y)| > t\})^{1/p} < \infty, \qquad 0 < p < \infty.$$

3. Interpolation Results

We begin this section by reformulating some of our previous results in terms of the approximation spaces $\bar{A}_{p,q;E}$ and then proceed to compute the K and E functionals for the $L^{p,q}(R^n)$ spaces with $0 < p, q \leqslant \infty$ and $BMO(R^n)$. In fact we treat the case $p = 0$ for the E functional as well. We start out by restating Corollary 2.6 as

Theorem 3.1. *Let* $0 < p < \infty$, $0 < q \leqslant \infty$. *Then*

$$(L^0, L^\infty)_{p,q;E} \approx (L^0, BMO)_{p,q;E} \approx L^{p,q}(R^n).$$

Proof. A well-known result of Peetre and Sparr [29] states that

$$E(t, f; L^0, L^\infty) = |\{y \in R^n : |f(y)| > t\}| = m(f, t). \tag{3.1}$$

In fact these authors show that $E(t, f; L^\infty, L^0) = f^*(t)$, which is equivalent to our statement since $E(t, f; L^0, L^\infty)^{-1} = E(t, f; L^\infty, L^0)$ (see the Gagliardo diagram in Section 1). Hence $(L^0, L^\infty)_{p,q;E} \approx L^{p,q}(R^n)$ is just one of the possible ways of defining the Lorentz spaces, as was done in (29). On the other hand using the trivial inequality

$$\|f\|_* \leqslant c\,\|f\|_\infty,$$

we readily see that

$$E(t, f; L^0, BMO) \leqslant E(ct, f; L^0, L^\infty)$$

and

$$\|f\|_{(L^0, BMO)_{p,q;E}} \leqslant c_1\,\|f\|_{(L^0, L^\infty)_{p,q;E}}.$$

Moreover, modulo constants and provided that $\alpha > 0$ is small enough, we have that

$$\|M^\#_{0,\alpha} f\|_0 \leqslant c\,\|f\|_0 \qquad \text{and} \qquad \|M^\#_{0,\alpha} f\|_\infty \leqslant c\,\|f\|_*.$$

Therefore we get that

$$|\{y \in R^n : M^\#_{0,\alpha} f(y) > t\}| \leqslant c_1 E(c_2 t, f; L^0, BMO). \tag{3.2}$$

Then Corollary 2.6 gives

$$\|f\|_{p,q} \leqslant c\,\|f\|_{(L^0,BMO)_{p,q;E}}$$

which completes the proof of the Theorem.

Theorem 3.1, in its equivalent formulation with the K-intermediate spaces, is essentially due to Hanks (15). Next we consider the question whether the estimate (3.2) is in fact an equivalence, as (3.1) suggests the case may be, but with $L^\infty(R^n)$ there replaced with $BMO(R^n)$. That this is the case is our next result.

THEOREM 3.2. *Let* $0<\alpha<(1000n^{1/2})^{-n}$. *Then there are constants* c_i, $1\leqslant i\leqslant 4$, *which depend on* α, *such that for* $t>0$,

$$\begin{aligned} c_1\,|\{y\in R^n\colon M^{\#}_{0,\alpha}f(y)>c_2t\}| &\leqslant E(t,f,L^0,BMO) \\ &\leqslant c_3\,|\{y\in R^n\colon M^{\#}_{0,\alpha}f(y)>c_4t\}|. \end{aligned}$$

Proof. We must only show the right-hand side inequality. We begin by describing what roughly amounts to be an optimal decomposition for the given function f in $L^0(R^n)+BMO(R^n)$. Let $\mathcal{O}$ be the open set of finite measure defined by $\mathcal{O}=\{y\in R^n\colon M^{\#}_{0,\alpha}f(y)>t\}$ and let $\{Q_j\}$ be a (dyadic) Whitney decomposition of $\mathcal{O}$. From (2.5) it is clear that with $\beta=10n^{1/2}$ each cube βQ_j contains some point x_j in $\mathcal{O}^c$. With $c_j=m_f(Q_j)$ we set

$$f_0=\sum_j (f-c_j)\,\chi_{Q_j} \qquad \text{and} \qquad f_1=\sum_j c_j\chi_{Q_j}+f\chi_{\mathcal{O}^c}.$$

Of course $f=f_0+f_1$ and $\|f_0\|_0\leqslant|\mathcal{O}|$. Invoking Corollary 2.3 it will suffice to prove that

$$\|M^{\#}_{0,1/4}f_1\|_\infty\leqslant ct.$$

For this purpose let Q be a fixed cube and let J be the collection of those indices j such that $|Q\cap Q_j|>0$. For c and C constants to be chosen appropriately, we estimate

$$\begin{aligned} |\{y\in Q\colon |f_1(y)-c|>Ct\}| &= \sum_{j\in J}|\{y\in Q\cap Q_j\colon |c_j-c|>Ct\}| \\ &\quad + |\{y\in Q\cap\mathcal{O}^c\colon |f(y)-c|>Ct\}|. \end{aligned}$$

We consider three mutually exclusive cases, to wit:

(1) $Q\subset\mathcal{O}^c$; in this case the first sum above vanishes. Also with the

choice $c=m_f(Q)$ we get, using (2.2) and the fact that $M^{\#}_{0,\alpha}f(y)\leqslant t$ for y in Q, that

$$|\{y\in Q: |f_1(y)-c|>10t\}|<\alpha\,|Q|<|Q|/4;$$

(2) $\operatorname{diam}(Q)\leqslant\operatorname{diam}(Q_{j_0})/5$ for some $j_0\in J$; by the properties of the Whitney decomposition it is clear that

(i) $Q\cap\mathcal{O}^c=\varnothing$,

(ii) $\operatorname{diam}(Q_j)/10\leqslant\operatorname{diam}(Q_{j_0})\leqslant 10\operatorname{diam}(Q_j)$, $j\in J$, and

(iii) $U_{j\in J}Q_j\subseteq 10Q_{j_0}$.

In this case we put $c=m_f(10\beta Q_{j_0})$. As a consequence of (ii) and (iii), as in the proof of Lemma 2.2, we get that $|c-c_j|\leqslant 20t$ for all j in J, provided that α is sufficiently small. Thus choosing $C=20$ because of (i) we see that

$$|\{y\in Q: |f_1(y)-c|>20t\}|=0;$$

(3) $\operatorname{diam}(Q)>\operatorname{diam}(Q_j)/5$ for all j in J; this time we have that $U_{j\in J}Q_j\subseteq 10Q$ and that the cube $10\beta Q$ contains some point $\bar{x}$ with $M^{\#}_{0,\alpha}f(\bar{x})\leqslant t$. By the triangle inequality

$$\begin{aligned}|\{y\in Q\cap Q_j: |c-c_j|>20t\}|\leqslant{}&|\{y\in Q\cap Q_j: |f(y)-c|>10t\}|\\&+|\{y\in Q\cap Q_j: |f(y)-c_j|>10t\}|,\end{aligned}$$

whence it follows that

$$\begin{aligned}|\{y\in Q: |f_1(y)-c|>20t\}|\leqslant{}&|\{y\in Q: |f(y)-c|>10t\}|\\&+\sum_{j\in J}|\{y\in Q_j: |f(y)-c_j|>10t\}|.\end{aligned}$$

Now set $c=m_f(10\beta Q)$. Replacing Q and Q_j by the larger sets $10\beta Q$ and $10\beta Q_j$, respectively, we get that

$$\begin{aligned}|\{y\in Q: |f_1(y)-c|>20t\}|&\leqslant\alpha\left(|10\beta Q|+\sum_{j\in J}|\beta Q_j|\right)\\&\leqslant 2\alpha\,|10\beta Q|<|Q|/4,\end{aligned}$$

because all the Q_j's are disjoint subsets of $10Q$ and $2\alpha(10\beta)^n<\frac{1}{4}$.

Altogether the above inequalities show that

$$\|M^{\#}_{0,1/4}f\|_\infty\leqslant 20t.$$

We conclude that

$$E(20\times 10^n t, f; L^0, BMO) \leqslant |\{y\in R^n: M^{\#}_{0,\alpha} f(y) > t\}|$$

which is what we wanted to show.

Combining Theorem 3.2 with general properties of intermediate spaces the reader can readily obtain several interesting corollaries. We single out three facts to discuss in detail, the first is the computation of the K-functionals.

COROLLARY 3.3. *Let $0<p<\infty$ and $0<q\leqslant\infty$. Then for $t>0$ and $f\in L^{p,q}(R^n)+BMO(R^n)$, we have that*

$$K(t, f; L^{p,q}, BMO) \approx \left(\int_0^{t^p} (s^{1/p}(M^{\#}_{0,\alpha} f)^*(s))^q \frac{ds}{s}\right)^{1/q} \qquad 0<q<\infty$$

and

$$K(t, f; L^{p,\infty}, BMO) \approx \sup_{0<s<t} s^{1/p}(M^{\#}_{0,\alpha} f)^*(s),$$

provided that $0<\alpha\leqslant\alpha_0=\alpha_0(n)$ is sufficiently small.

Proof. The general principle we will use is essentially the inverse of the better-known Holmstedt's formula (16) and states that for a pair $\bar{A}=(A_0, A_1)$ of normed Abelian groups,

$$K(t, a; \bar{A}_{p,q;E}, A_1) \approx \left(\int_0^{t^p} (s^{1/p}E(s, a; A_1, A_0))^q \frac{ds}{s}\right)^{1/q} \qquad 0<q<\infty \quad (3.3)$$

and a similar statement for $q=\infty$. The reader can consult Jawerth's paper (20), where, however, "our" E-functional is denoted by $\tilde{E}$. Now according to Theorem 3.2 we have that

$$E(s, f; BMO, L_0) = E(s, f; L_0, BMO)^{-1} \approx (M^{\#}_{0,\alpha} f)^*(s). \qquad (3.4)$$

Thus putting (3.4) in (3.3) yields

$$K(t, f; L^{p,q}, BMO) \approx \left(\int_0^{t^p} (s^{1/p}(M^{\#}_{0,\alpha} f)^*(s))^q \frac{ds}{s}\right)^{1/q} \qquad 0<q<\infty$$

(and a similar expression for the case $q=\infty$) since by Theorem 3.1 $L^{p,q}\approx (L^0, BMO)_{p,q;E}$. This completes our proof.

Other characterizations of the K-functional are easily obtained if we use an improvement of a result by Strömberg [37]. Set

$$M_p^{\#} f(x) = \sup_Q \inf_c \left(\frac{1}{|Q|} \int_Q |f(y) - c|^p \, dy \right)^{1/p}.$$

We then have

LEMMA 3.4. *Suppose that $0 < p < \infty$. Then for f in $L^p(R^n) + BMO(R^n)$ there are constants c_1, c_2 such that*

$$c_1 M_p M_{0,\alpha}^{\#} f(x) \leqslant M_p^{\#} f(x) \leqslant c_2 M_p M_{0,\alpha}^{\#} f(x)$$

provided that $0 < \alpha \leqslant \alpha_0(n)$ is small enough.

Proof. Fix an arbitrary cube Q_0 containing x_0. According to Corollary 2.6

$$\left(\frac{1}{|Q_0|} \int_{Q_0} |f(y) - m_f(Q)|^p \, dy \right)^{1/p} \leqslant c \left(\frac{1}{|Q_0|} \int_{Q_0} M_{0,\alpha}^{\#} f(y)^p \, dy \right)^{1/p}$$
$$\leqslant c M_p M_{0,\alpha}^{\#} f(x_0),$$

which readily gives

$$M_p^{\#} f(x_0) \leqslant c M_p M_{0,\alpha}^{\#} f(x_0).$$

For the converse, we fix again an arbitrary cube Q_0 containing x_0. Clearly

$$M_{0,\alpha}^{\#} f(x) \leqslant M_{0,\alpha;2Q_0}^{\#} f(x) + R_{0,\alpha;2Q_0} f(x),$$

where $R_{0,\alpha;NQ_0}$ is defined as $M_{0,\alpha}^{\#}$ except that the supremum is only taken over those cubes Q with $Q \cap (2Q_0)^c \neq \emptyset$.

Now

$$M_{0,\alpha;2Q_0}^{\#} f(x) \leqslant M_{0,\alpha;2Q_0}(f - c)(x)$$

for any constant c, and $R_{0,\alpha;2Q_0}$ is basically constant on Q_0; more precisely

$$\sup_{x \in Q_0} R_{0,\alpha;2Q_0} f(x) \leqslant \inf_{x \in Q_0} M_{0,\alpha'}^{\#} f(x)$$

with $\alpha' = \alpha(2/3)^n$. Hence, by (2.4)

$$\left(\frac{1}{|Q_0|} \int_{Q_0} M_{0,\alpha}^{\#} f(x)^p \, dx \right)^{1/p} \leqslant c \left(\frac{1}{|2Q_0|} \int_{2Q_0} |f(x) - c|^p \, dx \right)^{1/p}$$
$$+ M_{0,\alpha'}^{\#} f(x_0) \leqslant c M_p^{\#} f(x_0),$$

where the last inequality follows from Chebyshev's inequality as in the proof of Corollary 2.5. This completes the proof of the lemma.

The lemma is a limiting case of the statement that

$$M_p M_{p'}^{\#} f(x) \approx M_p^{\#} f(x)$$

if $0 < p' < p < \infty$, which can be proved in a similar way. In particular combining the lemma with Corollary 3.3 we get the following result, due to Bennett and Sharpley [4].

COROLLARY 3.5. *Let $0 < p < \infty$. Then for $t > 0$ and f in $L^p(R^n) + BMO(R^n)$ we have that*

$$K(t, f; L^p, BMO) \approx t(M_p^{\#} f)^*(t^p).$$

Remark 3.6. It is also easy to see that

$$\begin{aligned} c_1 \, |\{ y \in R^n : M_p M_{0,\alpha}^{\#} f(y) > c_2 t \}| &\leqslant E(t, f; L^p, BMO) \\ &\leqslant c_3 \, |\{ y \in R^n : M_p M_{0,\alpha}^{\#} f(y) > c_4 t \}| \end{aligned}$$

again by using Theorem 3.1 and general properties of interpolation spaces (or by taking inverses in Corollary 3.3). Clearly c_1, c_2, c_3, c_4 depend on α.

Remark 3.7. It is apparent that the results of this section hold, with minor changes, in the weighted case as well, provided that the weight w satisfies a doubling condition. For instance, it may be shown that $E(t, f; L^0(w\,dx), BMO(w))$ is, up to equivalence $w(\{M_{0,\alpha;w}^{\#} f > t\})$ if α is small enough (cf. Remark 2.11). This result leads, as we saw at the beginning of this section, to $K(t, f; L^{p,q}(w), BMO(w)) \approx (\int_0^{t^p} (s^{1/p} [w(\{M_{0,\alpha;w}^{\#} f > t\})]^{-1}(s))^q \, ds/s)^{1/q}$, $q < \infty$, and a similar expression when $q = \infty$.

We close this chapter with an extension of a result of Rivière [30].

THEOREM 3.8. *Suppose that T is a subadditive operator which maps $L^{p_0,r_0}(R^n)$ into $L^{q_0,s_0}(R^n)$ and $L^{p_1,r_1}(R^n)$ into $BMO(R^n)$ continuously, where $0 \leqslant p_0 < p_1 \leqslant \infty$, $0 \leqslant q_0 < \infty$, $0 < r_0$, r_1, $s_0 \leqslant \infty$. Then if $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0$, for $0 < \theta < 1$, and $0 < r \leqslant s \leqslant \infty$, we have that T maps $L^{p,r}(R^n)$ into $L^{q,s}(R^n)$ (modulo constants) continuously and there is a constant c which depends only on n and the norm of the operator in the given spaces such that*

$$\|Tf\|_{q,s} \leqslant c \, \|f\|_{p,r}.$$

Proof. It is readily seen that the assumptions imply (and are in fact equivalent to)

$$E(t, \mathrm{Tf}; L^{q_0,s_0}, BMO) \leqslant c_1 E(c_2 t, f; L^{p_0,r_0}, L^{p_1,r_1}).$$

The details needed now to complete the proof, being immediate, are left for the interested reader to provide.

4. Integral Operators

Rivière's interpolation theorem in the particular case of subadditive operators mapping continuously $L^1(R^n)$ into $L^{1,\infty}(R^n)$, or of weak-type $(1, 1)$, and $L^\infty(R^n)$ into $BMO(R^n)$, or of type $(\infty, *)$, has from the classical point of view some of the most important applications. In this context see also Spanne [32] and Stampacchia [33]. We look therefore more closely to that situation and give in Propositions 4.1 and 4.3 two characterizations of such operators. We then consider, more in detail, integral operators T with a structure that, roughly speaking, makes them look more like the Hardy–Littlewood maximal function. For these operators we are able to derive pointwise estimates involving maximal functions in the spirit of the ones obtained by Córdoba and Fefferman [9] but preserving the weak-type information. We close the chapter with applications of these results to weighted and vector-valued inequalities.

Let us then begin by considering a subadditive operator T such that

$$T\colon\ L^1(R^n) \to L^{1,\infty}(R^n); \qquad T\colon\ L^\infty(R^n) \to BMO(R^n). \tag{4.1}$$

Of course (4.1) implies that for f in $L^1(R^n) + L^\infty(R^n)$ and all $t > 0$

$$K(t, \mathrm{Tf}; L^{1,\infty}, BMO) \leqslant cK(t, f; L^1, L^\infty). \tag{4.2}$$

However, by Theorem 3, $K(t, \mathrm{Tf}; L^{1,\infty}, BMO) \approx \sup_{0<s<t} s(M^{\#}_{0,\alpha}\mathrm{Tf})^*(s)$, provided that α is small enough, and it is well known that $K(t, f; L^1, L^\infty) = \int_0^t f^*(s)\, ds$. Hence we may rewrite (4.2) as

$$\sup_{0<s<t} s(M^{\#}_{0,\alpha}\mathrm{Tf})^*(s) \leqslant c \int_0^t f^*(s)\, ds. \tag{4.3}$$

By letting t tend to ∞, we observe that (4.3) implies that T is of weak-type $(1, 1)$, modulo constants, and by first setting $s = t$ in the left-hand side, dividing by t and then letting t tend to 0, Eq. (4.3) also gives that T is of type $(\infty, *)$. Equations (4.1) and (4.3) are thus equivalent. Moreover, since $t(Mf)^*(t) \approx \int_0^t f^*(s)\, ds$, by taking inverses in (4.3), we obtain

PROPOSITION 4.1. *A subadditive operator T is of weak-type* $(1, 1)$ *and maps $L^\infty(R^n)$ into $BMO(R^n)$ continuously if and only if for every f in $L^1(R^n)+L^\infty(R^n)$ and $t>0$,*

$$|\{y\in R^n: M^{\#}_{0,\alpha}\mathrm{Tf}(y)>t\}|\leqslant c_1\,|\{y\in R^n: Mf(y)>c_2 t\}|$$

for some constants c_1, c_2 depending on n, α, T, provided $\alpha\leqslant\alpha_0(n)$ is sufficiently small.

Proposition 4.1 is the prototype of statements involving Lorentz spaces, weighted spaces and other spaces of interest in harmonic analysis. Using the known K-functionals in the various cases the reader is invited to provide the statements of the results analogous to Proposition 4.1 in these settings; the following case which may serve as another illustration of this principle is also of interest to us.

PROPOSITION 4.2. *A subadditive operator T is of weak-type* $(1, 1)$ *and maps $BMO(R^n)$ continuously into itself if and only if for every f in $L^1(R^n)+BMO(R^n)$ and $t>0$*

$$|\{y\in R^n: M^{\#}_{0,\alpha}\mathrm{Tf}(y)>t\}|\leqslant c_1\,|\{y\in R^n: M^{\#}f(y)>c_2 t\}|$$

for constants c_1, c_2 depending only on n, α, T, provided that $\alpha\leqslant\alpha_0(n)$ is sufficiently small.

Proof. Similar to that of Proposition 4.1.

The second characterization of operators satisfying (4.1) is given by

PROPOSITION 4.3. *A subadditive operator T is of weak-type* $(1, 1)$ *and maps $L^\infty(R^n)$ into $BMO(R^n)$ continuously if and only if for every cube Q in R^n, f in $L^1(R^n)+L^\infty(R^n)$ and $t>0$,*

$$|\{y\in Q: |\mathrm{Tf}(y)-m_{\mathrm{Tf}}(Q)|>t\}|\leqslant c_1 e^{-c_2 t/\|f\|_\infty}\min(|Q|, \|f\|_1/t)$$

for some constants c_1, c_2 depending only on n and T.

Proof. First suppose that T satisfies (4.1) and that f is essentially bounded. Then

$$\begin{aligned}&|\{y\in Q: |\mathrm{Tf}(y)-m_{\mathrm{Tf}}(Q)|>t\}|\\&\quad\leqslant|\{y\in Q: |\mathrm{Tf}(y)-m_{\mathrm{Tf}}(Q)|>t,\ M^{\#}_{0,\alpha}\mathrm{Tf}(y)\leqslant c\,\|f\|_\infty\}|\\&\qquad+|\{y\in Q: M^{\#}_{0,\alpha}\mathrm{Tf}(y)>c\,\|f\|_\infty\}|.\end{aligned}$$

The second summand in the right-hand side vanishes if c is sufficiently large. As for the first summand we apply Theorem 2.7 with $\beta=c\,\|f\|_\infty/t$

and $p=0$ and also $p=1$ in case $f \in L^1(R^n)$, otherwise the estimate involving $\|f\|_1$ is obvious. If $f \notin L^\infty(R^n)$ the estimates are equally trivial. This proves the necessity of the condition.

To show the sufficiency we begin by observing that the statement concerning the type $(\infty, *)$ is immediate. Suppose now that f is integrable and let $\{Q_j\}_{j=1}^\infty$ be a sequence of cubes increasing to R^n. We claim that $\lim_{j \to \infty} m_{\mathrm{Tf}}(Q_j) = c_{\mathrm{Tf}}$ exists. Indeed, let $0<\varepsilon<1$ be fixed. It will suffice to show that there is a sufficiently large index N such that for $j, k > N$,

$$|m_{\mathrm{Tf}}(Q_j) - m_{\mathrm{Tf}}(Q_k)| \leqslant \varepsilon. \tag{4.4}$$

Let $j<k$. Then

$$\begin{aligned} I &= |\{y \in Q_j\colon |m_{\mathrm{Tf}}(Q_j) - m_{\mathrm{Tf}}(Q_k)| > \varepsilon\}| \\ &\leqslant |\{y \in Q_j\colon |\mathrm{Tf}(y) - m_{\mathrm{Tf}}(Q_j)| > \varepsilon/2\}| \\ &\quad + |\{y \in Q_k\colon |\mathrm{Tf}(y) - m_{\mathrm{Tf}}(Q_k)| > \varepsilon/2\}| \\ &\leqslant 2c_1 e^{-c_2\varepsilon/2\|f\|_\infty} \|f\|_1/\varepsilon. \end{aligned}$$

Since I equals either $|Q_j|$ or zero and $|Q_j|$ tends to ∞ with j, we readily see that (4.4) holds provided j is large enough. If Q is now so large that $|c_{\mathrm{Tf}} - m_{\mathrm{Tf}}(Q)| < t/2$, then

$$\begin{aligned} |\{y \in Q\colon |\mathrm{Tf}(y) - c_{\mathrm{Tf}}| > t\}| &\leqslant |\{y \in Q\colon |\mathrm{Tf}(y) - m_{\mathrm{Tf}}(Q)| > t/2\}| \\ &\leqslant c_1 e^{-c_2 t/\|f\|_\infty} \|f\|_1/t. \end{aligned} \tag{4.5}$$

As Q is arbitrary, we get that T is of weak-type $(1, 1)$ (modulo constants). This completes the proof.

Remark 4.4. The estimate (4.5) for T, a singular integral operator, is due to Rivière.

Proposition 4.3 admits a more localized version, namely

PROPOSITION 4.5. *Suppose S and T are operators such that $M^{\#}_{0,\alpha}\mathrm{Tf}(x) \leqslant cSf(x)$, for α sufficiently small, and S is of weak-type* $(1, 1)$. *Then there is a constant c_{Tf} such that*

$$|\{y \in R^n\colon |\mathrm{Tf}(y) - c_{\mathrm{Tf}}| > t,\ Sf(y) \leqslant \lambda t\}| \leqslant c_1 e^{-c_2/\lambda} \|f\|_1/t$$

for constants c_1, c_2 depending only on n, α, S, and T.

The proof, which follows at once from Theorem 2.7, is left to the reader. The formulation of Proposition 4.5 in case of finite cubes Q instead of R^n is also left to the interested reader.

To apply Proposition 4.5, it becomes important to determine under what conditions we can find a weak-type (1, 1) operator S so that

$$M^{\#}_{0,\alpha}\mathrm{Tf}(x) \leqslant cSf(x). \tag{4.6}$$

We shall see, as a particular case of Example 4.9, that when $n=1$ and $T=$ Hilbert transform, we can choose $S=M$, the Hardy–Littlewood maximal function. In this case Proposition 4.5 becomes a recent result of Muckenhoupt [27], who improved on an earlier result due to Hunt [18]. Hunt used his version of Proposition 4.5 to show that

$$T^{*}_{\{n_k\}}f(x) = \sup_k |S_{n_k}f(x)|/\log\log n_k < \infty,$$

a.e. for all $f \in L^1([-\pi, \pi))$. Here $S_{n_k}f(x)$ denotes the n_kth partial sum of the Fourier series of f for the lacunary sequence $\{n_k\}$. Then by employing a theorem of Stein [34], the weak-type (1, 1) estimate for $T^{*}_{\{n_k\}}f$ is obtained. The application of Proposition 4.5 avoids the use of [34]. Moreover we can show, by means of an argument similar to the one we shall employ in the proof of Example 4.9, that in this case we have inequality (4.6) with $T=T^{*}_{\{n_k\}}$ and $S=M^{\#}$, the sharp-maximal operator.

Let us consider, then, the possibility of improving the "control in probability" given by Propositions 4.1 and 4.2 to pointwise estimates such as (4.6), with $S=M$ or $M^{\#}$ such as in the instances described above. To this end we introduce the following conditions pertaining to kernels $k(x, y)$ defined on $R^n \times R^n \backslash$diagonal,

$$(A_{\Phi}) \qquad \sup_x \sup_\delta \int_{|u|\leqslant 1}\int_{|v|\leqslant 1} |k_\delta(x+u, x+y) - k_\delta(x+v, x+y)|\,du\,dv \leqslant \Phi(y)$$

and

$$(A'_{\Phi}) \qquad \sup_\delta \int_{|u|\leqslant 1}\int_{|v|\leqslant 1} |k_\delta(u-y) - k_\delta(v-y)|\,du\,dv \leqslant \Phi(y),$$

where as usual $k_\delta(x, y) = \delta^{-n}k(x/\delta, y/\delta)$ and $k_\delta(y) = \delta^{-n}k(y/\delta)$. We then have

THEOREM 4.6. *Let* $\mathrm{Tf}(x) = \int_{R^n} k(x, y)\, f(y)\, dy$ *be an integral operator of weak-type* (1, 1) *and suppose that* $k(x, y)$ *satisfies* (A_Φ), *where for* $|z| \geqslant N$, *some large value,* $\Phi(z)$ *is a radial, nonincreasing, summable function. Then for* $\alpha \leqslant \alpha_0(n)$ *sufficiently small we have with* $c = c(\alpha)$,

$$M^{\#}_{0,\alpha}Tf(x) \leqslant cMf(x).$$

Proof. For convenience we introduce the "centered maximal functions"

$$\tilde{M}f(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

x = center of Q, and similarly for $\tilde{M}^{\#}_{0,\alpha} f(x)$. It is readily seen that $Mf(x) \approx \tilde{M}f(x)$ and similarly for $\tilde{M}^{\#}_{0,\alpha} f(x)$.

Let Q be a cube centered at x_0. For a fixed locally summable function f we set

$$f(x) = f(x)\, \chi_{n^{1/2}NQ}(x) + f(x)\, \chi_{(n^{1/2}NQ)^c}(x) = f_1(x) + f_2(x), \qquad \text{say.}$$

Since T is of weak-type $(1, 1)$ we have

$$\begin{aligned} |\{y \in Q \colon |Tf_1(y)| > t\}| &\leqslant |\{y \in R^n \colon |Tf_1(y)| > t\}| \\ &\leqslant c\, \|f_1\|_1/t \leqslant c\, |n^{1/2}NQ|\, \tilde{M}f(x_0)/t \leqslant \alpha\, |Q|/2, \end{aligned} \tag{4.7}$$

provided that $t > c'\tilde{M}f(x_0)$.

As for f_2, by Chebychev's inequality we have

$$\begin{aligned} |\{y \in Q \colon |Tf_2(y) - (Tf_2)_Q| > t\}| &\leqslant \int_Q |Tf_2(y) - (Tf_2)_Q| \, dy/t \\ &\leqslant \int |f_2(y)| \left(\frac{1}{|Q|} \int_Q \int_Q |k(x, y) - k(z, y)| \, dx\, dz \right) dy/t. \end{aligned} \tag{4.8}$$

Let us estimate the innermost integrals I. The cube Q is obviously contained in a ball with the same center as Q and with radius $\delta = \operatorname{diam} Q/2 = n^{1/2}\, |Q|^{1/n}/2$. Hence, by changing variables we see that

$$\begin{aligned} I &\leqslant \frac{\delta^{2n}}{|Q|} \int_{|u| \leqslant 1} \int_{|v| \leqslant 1} |k(\delta u + x_0, y) - k(\delta v + x_0, y)| \, du\, dv \\ &= \frac{\delta^n}{|Q|} \int_{|u| \leqslant 1} \int_{|v| \leqslant 1} |k_{\delta^{-1}}(u + x_0/\delta, (y - x_0)/\delta + x_0/\delta) \\ &\quad - k_{\delta^{-1}}(v + x_0/\delta, (y - x_0)/\delta + x_0/\delta)| \, du\, dv. \end{aligned}$$

Thus by (A_Φ),

$$I \leqslant \frac{\delta^n}{|Q|} \Phi((y - x_0)/\delta) = \frac{\delta^{2n}}{|Q|} \Phi_\delta(y - x_0) \leqslant c\, |Q|\, \Phi_\delta(y - x_0)$$

whence the right-hand side of (4.8) is majorized by

$$c\, |Q| \int |f_2(y)|\, \Phi_\delta(x_0 - y)\, dy \tag{4.9}$$

which in turn does not exceed

$$c\,|Q|\,\tilde{M}f(x_0)/t \tag{4.10}$$

since by our assumptions $\sup_{\delta>0} |f_2| * \Phi_\delta(x_0) \leqslant c\tilde{M}f(x_0)$. Thus

$$|\{y \in Q: |\mathrm{Tf}_2(y) - (\mathrm{Tf}_2)_Q| > t\}| \leqslant c\tilde{M}f(x_0)\,|Q|/t < \alpha\,|Q|/2 \tag{4.11}$$

provided that $t > c\tilde{M}f(x_0)$. Since Q is arbitrary we can combine (4.8) and (4.11) and obtain that for all x in R^n,

$$\tilde{M}^{\#}_{0,\alpha}\mathrm{Tf}(x) \leqslant c\tilde{M}f(x),$$

which is equivalent to the desired conclusion.

In the same spirit we prove

THEOREM 4.7. *Let* $\mathrm{Tf}(x) = \int_{R^n} k(x, y)\, f(y)\, dy$ *be an integral operator of weak-type* $(1, 1)$ *and suppose that* $k(x, y)$ *satisfies* (A_Φ) *where for* $|z| \geqslant N$, $\Phi(z)$ *is a radial, nonincreasing function such that* $\int_{|z| \geqslant N} \Phi(z) \log |z|\, dz < \infty$. *Then for* $\alpha \leqslant \alpha_0(n)$ *sufficiently small we have*

$$M^{\#}_{0,\alpha}\mathrm{Tf}(x) \leqslant cM^{\#}f(x).$$

Proof. Let $\tilde{M}^{\#}f(x)$ denote the centered sharp maximal function and let Q be a cube centered at x_0. Since $\tilde{M}^{\#}f(x) = \tilde{M}^{\#}(f-c)(x)$ for any constant c, there is no loss in generality in assuming that $f_{n^{1/2}NQ} = (1/|n^{1/2}NQ|) \int_{n^{1/2}NQ} f(y)\, dy = 0$. The argument used in the proof of Theorem 4.6 needs only minor changes once the following variant of Lemma 2.4 of Fefferman and Stein [12] is invoked.

LEMMA 4.8. *For a given cube* Q *and* $a > 1$,

$$|f_{a^jQ} - f_Q| \leqslant cj \inf_{x \in Q} M^{\#}f(x), \qquad j = 1, 2, \ldots.$$

The proof of this lemma, being immediate, is not given here. To continue with the proof of the theorem we decompose $f = f_1 + f_2$ as in Theorem 4.6. Since $f_{n^{1/2}NQ} = 0$ we may treat Tf_1 exactly as before. To estimate the term involving Tf_2 we consider, in view of (4.8) and (4.9),

$$I = \int |f_2(y)|\, \Phi_\delta(x_0 - y)\, dy \leqslant \sum_{j=0}^{\infty} \int_{b_{j+1}Q \setminus b_jQ} |f(y)|\, \Phi_\delta(x_0 - y)\, dy$$

$$\leqslant \sum_{j=0}^{\infty} \Phi_\delta(b_j\,|Q|^{1/n}/2) \int_{b_{j+1}Q} |f(y)|\, dy,$$

with $\delta = n^{1/2}\,|Q|^{1/n}/2$ and $b_j = 2^j n^{1/2} N$. This is clearly less than

$$c \sum_{j=0}^{\infty} \Phi(2^j N) \frac{1}{|b_{j+1}Q|} \int_{b_{j+1}Q} |f(y) - f_{b_{j+1}Q}|\, dy$$

$$+ c \sum_{j=0}^{\infty} \Phi(2^j N)\, |f_{b_{j+1}Q} - f_{b_0 Q}| = cI_1 + cI_2,$$

say. As the integral in each of the summands of I_1 does not exceed $|b_{j+1}Q|\, \tilde{M}^{\#} f(x_0)$, it follows that

$$I_1 \leqslant c\left(\sum_{j=0}^{\infty} \Phi(2^j N)\right) \tilde{M}^{\#} f(x_0) \leqslant c\left(\int_{|y| > N/2} \Phi(y)\, dy\right) \tilde{M}^{\#} f(x_0).$$

Moreover, according to Lemma 4.8,

$$I_2 \leqslant c\left(\sum_{j=0}^{\infty} \Phi(2^j N)\, j\right) \tilde{M}^{\#} f(x_0)$$

$$\leqslant c\left(\int_{|y| > N/2} \Phi(y) \log |y|\, dy\right) \tilde{M}^{\#} f(x_0).$$

Putting these estimates together we get that

$$I \leqslant c\tilde{M}^{\#} f(x_0)$$

and consequently

$$|\{y \in Q : |\mathrm{Tf}_2(y) - (\mathrm{Tf}_2)_Q| > t\}| \leqslant c\tilde{M}^{\#} f(x_0)\, |Q|/t < \alpha\, |Q|/2,$$

provided that $t > c\tilde{M}^{\#} f(x_0)$. Since Q is arbitrary we conclude that

$$\tilde{M}^{\#}_{0,\alpha} \mathrm{Tf}(x) \leqslant c\tilde{M}^{\#} f(x) \qquad \text{all } x \text{ in } R^n.$$

This is what we wanted to show.

Remark 4.9. It is readily seen that Propositions 4.1 and 4.2 and Theorems 4.6 and 4.7 remain true if we replace $M^{\#}_{0,\alpha}\mathrm{Tf}$ by $M^{\#}_p\mathrm{Tf}$ for $0 < p < 1$.

We discuss now some examples of kernels which satisfy (A'_Φ), and consequently (A_Φ).

EXAMPLE 4.10. Let $k(x) = \Omega(x)/|x|^n$ be a classical Calderón–Zygmund kernel with $\Omega(x)$ a homogeneous function of degree 0 and $\int_\Sigma \Omega(x')$

$d\sigma(x')=0$. We also assume that Ω is essentially bounded on Σ and that it satisfies the Dini condition $\int_0^1 w(\delta)\, d\delta/\delta < \infty$, where

$$w(\delta)=\sup_{|y|>1/\delta}\int_{|u|\leqslant 1}\int_{|v|\leqslant 1}|\Omega(u-y)-\Omega(v-y)|\,du\,dv.$$

For these kernels we have (A'_Φ) with $\Phi(y)=cw(c/|y|)\,|y|^{-n}+c\,\|\Omega\|_{L^\infty(\Sigma)}\,|y|^{-(n+1)}$. Indeed, because of the homogeneity of the kernel it is enough to consider $\delta=1$. In that case and for $|y|>10$, we have

$$\begin{aligned}\int_{|u|\leqslant 1}\int_{|v|\leqslant 1}&|k(u-y)-k(v-y)|\,du\,dv\\ &\leqslant\int_{|u|\leqslant 1}\int_{|v|\leqslant 1}\frac{|\Omega(u-y)-\Omega(v-y)|}{|u-y|^n}\,du\,dv\\ &\quad+\|\Omega\|_{L^\infty(\Sigma)}\int_{|u|\leqslant 1}\int_{|v|\leqslant 1}\left|\frac{1}{|u-y|^n}-\frac{1}{|v-y|^n}\right|du\,dv\\ &\leqslant cw(c/|y|)\,|y|^{-n}+c\,\|\Omega\|_{L^\infty(\Sigma)}\,|y|^{-(n+1)}=\Phi(y).\end{aligned}$$

As the singular integral operator $\mathrm{Tf}(x)=\text{p.v.}\int_{R^n}k(x-y)\,f(y)\,dy$ is of weak-type $(1, 1)$, cf. [6], from Theorem 4.6 it follows that

$$M^{\#}_{0,\alpha}\,\mathrm{Tf}(x)\leqslant c\,\mathrm{Mf}(x).$$

If on the other hand $\int_0^1 w(\delta)\log(e/\delta)\,d\delta/\delta<\infty$, by Theorem 4.7, we conclude that

$$M^{\#}_{0,\alpha}\,\mathrm{Tf}(x)\leqslant cM^{\#}f(x).$$

In the next two examples we assume the condition (A_Φ).

Example 4.11. Assume that the kernel $k(x, y)$ satisfies the (A_Φ) condition with Φ radial, nonincreasing, integrable (for large values) and that $|k(x, y)|\leqslant c\,|x-y|^{-n}$. Let $k^\varepsilon(x, y)=k(x, y)$ when $|x-y|\geqslant\varepsilon$ and 0 otherwise. If the operator

$$T^*f(x)=\sup_{\varepsilon>0}\left|\int k^\varepsilon(x, y)\,f(y)\,dy\right|$$

is of weak-type $(1, 1)$, then with $c=c(\alpha)$,

$$M^{\#}_{0,\alpha}\,T^*f(x)\leqslant c\,\mathrm{Mf}(x).$$

The proof is immediate. For a fixed cube Q centered at x_0 and a given f

we put $f = f_1 + f_2$, with f_1 supported in a multiple of Q and f_2 supported in the complement. For f_1 we see at once that

$$\tilde{M}^{\#}_{0,\alpha} T^* f_1(x_0) \leq c\tilde{M} f(x_0).$$

As for f_2 observe that for $x \in Q$,

$$\begin{aligned} &|T^* f_2(x) - (T^* f_2)_Q| \\ &\quad \leq \sup_{\varepsilon > 0} \left| \int_Q \left| \int (k^\varepsilon(x, y) - k^\varepsilon(z, y)) f_2(y)\, dy \right| dz \right| \\ &\quad \leq c \left(\tilde{M} f(x_0) + \int |f_2(y)| \left(\int_Q |k(x, y) - k(z, y)|\, dz \right) dy \right). \end{aligned}$$

By the argument used to bound the similar expression (4.8) it follows that

$$\tilde{M}^{\#}_{0,\alpha} T^* f_2(x_0) \leq c\tilde{M} f(x_0)$$

This completes our discussion. The weak-type (1, 1) estimate for T^* is known to hold, for instance, for Calderón–Zygmund operators, cf. [6].

EXAMPLE 4.12. We say that a function $p(x, \xi)$ is a classical symbol in the class $S^m_{\rho,\delta}$ provided that

$$|\partial^\beta_x \partial^\gamma_\xi p(x, \xi)| \leq c_{\beta,\gamma}(1 + |\xi|)^{m - \rho|\gamma| + \delta|\beta|}$$

for all multi-indices β, γ, and all x, ξ in R^n. Consider the pseudodifferential operator $(\psi.d.o)$

$$p(x, D) f(x) = \int_{R^n} e^{2\pi i(x,\xi)} p(x, \xi) \hat{f}(\xi)\, d\xi$$

defined a priori for Schwartz functions f on R^n. We will sketch the proof that the $\psi.d.o$ $p(x, D)$ defines an integral operator with kernel satisfying the (A_Φ) condition with $\Phi(z) = |z|^{-(n+\varepsilon)}$, with $\varepsilon = \frac{1}{2}$, for instance, for $|z|$ sufficiently large, when $p(x, \xi)$ is in the class $S^0_{1,\delta}$, $0 < \delta < 1$.

For this purpose let ϕ be a function supported in $\{\xi : 2^{-1} \leq |\xi| \leq 2\}$ and such that $\sum_{\nu=-\infty}^{\infty} \phi(2^{-\nu}\xi) = 1$, $\xi \neq 0$. Let $p_\nu(x, D)$ denote the $\psi.d.o$ with symbol $p_\nu(x, \xi) = p(x, \xi)\, \phi(2^{-\nu}\xi)$. We can write

$$p_\nu(x, D) f(x) = \int_{R^n} k_\nu(x, y) f(y)\, dy,$$

where the kernel k_ν is defined by

$$k_\nu(x, y) = \int_{R^n} e^{2\pi i(x-y,\xi)} p_\nu(x, \xi)\, d\xi.$$

The estimate we need is given by

LEMMA 4.13. *Let $p(x, \xi)$ be a symbol of class $S^m_{1,\delta}$, $m > 0$. For every $a \geqslant m$ we have*

$$|k_\nu(x, y)| \leqslant c\, |x-y|^{-a} \min(2^{\nu n}, 2^{\nu(n+m-a)}).$$

Proof. It is left to the reader with the observation that when a is an integer it follows directly by partial integration and in the general case it follows from this case at once.

We return to the proof. Since p is a symbol of class $S^0_{1,\delta}$ so is p_ν, and moreover from Lemma 4.13 it readily follows that the kernel k_ν satisfies the estimate

$$t^{-n}\, |k_\nu(x/t, y/t)| \leqslant ct\, |z-y|^{-(n+1)} \min(2^{\nu n}, 2^{-\nu}) \tag{4.12}$$

whenever $|x-z| \leqslant 1$ and $|y| \geqslant 10$. On the other hand for $x = (x_1,..., x_n)$, and $z = (z_1,..., z_n)$

$$k_\nu(x, y) - k_\nu(z, y) = \sum_{j=1}^{n} (x_j - z_j) \int_{R^n} \int_0^1 e^{2\pi i(x(s)-y,\xi)}$$
$$\times \left(\frac{\partial p_\nu}{\partial x_j}(x(s), \xi) + 2\pi i \xi_j\, p_\nu(x(s), \xi) \right) ds\, d\xi,$$

where $x(s) = z + s(x-z)$. We have two different kinds of terms in the above sum. For the terms involving $\partial p_\nu/\partial x_j$ we use Lemma 4.13 with $m = \delta$, $a = 2n$, and for the terms involving $\xi_j p_\nu$ we use it with $m = 1$, $a = n + \frac{1}{2}$ to get that

$$\begin{aligned} t^{-n}\, &|k_\nu(x/t, y/t) - k_\nu(z/t, y/t)| \\ &\leqslant c\{t^n\, |y-z|^{-2n} \min(2^{\nu n}, 2^{-\nu(1-\delta)}) \\ &\quad + t^{1/2}\, |y-z|^{-(n+1/2)} \min(2^{\nu n}, 2^{\nu/2})\}. \end{aligned} \tag{4.13}$$

Finally to show that the kernel k of the $\psi.d.o$ p satisfies the desired (A_Φ) estimate we put $k = \sum_{\nu=-\infty}^{\infty} k_\nu$ and invoke (4.12) for the expressions involving large ν's, namely whenever $2^\nu \delta \geqslant 1$, and (4.13) for the remaining ν's. The desired conclusion follows upon summing over ν. Illner [19] has shown that the operators $p(x, D)$ with symbols in $S^0_{1,\delta}$ are of weak-type (1, 1). From Theorem 4.6 we conclude at once that

$$M^{\#}_{0,\alpha}(p(\cdot, D)f)(x) \leqslant cM^{\#}f(x) \tag{4.14}$$

provided α is sufficiently small.

EXAMPLE 4.14. This example is related to results of Aguilera [1]. For simplicity in the notation, we assume that $n = 2$. Set

$$M_k f(x) = \sup_{x \in R} \frac{1}{|R|} \int_R |f(y)|\, dy,$$

where $\{R\}$ is the family of all rectangles with sides parallel to the coordinate axes such that the ratio longer side–shorter side $= 2^k$. Let $\mathscr{F}$ denote the family of all rectangles centered at the origin with sides parallel to the coordinate axes and let

$$T_R f(x) = \int_{R^n \setminus R} k(y)\, f(x-y)\, dy, \qquad R \in \mathscr{F},$$

where $|k(y)| \leqslant c\, |y|^{-2}$, k satisfies (A_Φ) with Φ radial, nonincreasing and integrable and the operator

$$Tf(x) = (p.v) \int k(y)\, f(x-y)\, dy$$

is of weak-type (1, 1). Then for

$$T^*_{\mathscr{F}} f(x) = \sup_{R \in \mathscr{F}} |T_R f(x)|,$$

it follows as in [1] that $T^*_{\mathscr{F}}$ is of weak-type (1, 1). We now claim that

$$M^{\#}_{0,\alpha}(T^*_{\mathscr{F}} f)(x) \leqslant c \sum_{k=0}^{\infty} 2^{-k} M_k f(x). \tag{4.15}$$

To prove (4.15) fix a cube Q centered at x_0. As in Example 4.11 we estimate

$$\sup_{R \in \mathscr{F}} |T_R f(x) - T_R f(z)|,$$

where $\operatorname{supp} f \subseteq (\beta Q)^c$, $\beta = n^{1/2}N$ and x, z are in Q. Let $R = \{x = (x_1, x_2): |x_1| \leqslant a, |x_2| \leqslant b\}$ be an arbitrary rectangle with $a \geqslant b$, say. Clearly

$$|T_R f(x) - T_R f(z)| \leqslant c \int |\chi_R(x-y)\, k(x-y) - \chi_R(z-y)\, k(z-y)|\, |f(y)|\, dy$$
$$= cI.$$

Having fixed x and z the values of y for which the above integral does not vanish fall into three classes, namely: (i) both $\chi_R(x-y)$ and $\chi_R(z-y)$ are 1, (ii) both $\chi_R(x-y)$ and $\chi_R(z-y)$ are 0, and (iii) one of the characteristic functions is 1 and the other is 0. In case (i) we get the bound

$$I \leqslant \int |k(x-y) - k(z-y)|\, |f(y)|\, dy$$

and as in Theorem 4.6 we see that in fact

$$I \leqslant c\tilde{M}f(x_0).$$

In case (ii) there is nothing to prove as $I = 0$. In case (iii) to fix ideas suppose that $\chi_R(x-y) = 1$ and $\chi_R(z-y) = 0$. This means that $|x_1 - y_1| < a$, $|x_2 - y_2| < b$ and in addition one of the following three conditions holds, to wit: (iii$_1$) $|z_1 - y_1| < a$, $|z_2 - y_2| > b$, (iii$_2$) $|z_1 - y_1| > a$, $|z_2 - y_2| > b$, or (iii$_3$) $|z_1 - y_1| > a$, $|z_2 - y_2| < b$. Because of our assumptions, since $x \in Q$ and $\operatorname{supp} f \subseteq (\beta Q)^c$ we have that $|x_2 - y_2| \approx |z_2 - y_2|$. Hence if either (iii$_1$) or (iii$_2$) holds, then $|x_2 - y_2| \approx b$. Let $I_0 = \{y = (y_1, y_2): |x_1 - y_2| < b, |x_2 - y_2| < b\}$ and $I_k = \{y = (y_1, y_2): 2^{k-1}b < |x_1 - y_1| < 2^k b, |x_2 - y_2| < b\}$, $k \geqslant 1$. Then

$$\begin{aligned} I &\leqslant c \sum_{k=0}^{\infty} \int \chi_{I_k}(y) \max(|x_1 - y_1|, |x_2 - y_2|)^{-2}\, |f(y)|\, dy \\ &\leqslant cb^{-2}\left(\int \chi_{I_0}(y)\, |f(y)|\, dy + \sum_{k=1}^{\infty} 2^{-2k} \int_{|x_1 - y_1| < 2^{k+1}b, |x_2 - y_2| < b} |f(y)|\, dy \right) \\ &\leqslant c \sum_{k=0}^{\infty} 2^{-k} \tilde{M}_k f(x_0). \end{aligned}$$

As case (iii$_3$) can be handled in a similar way our proof is complete.

The main application of the point-wise estimates discussed above is to weighted inequalities, including the limiting case of weak-type (1, 1), and to vector-valued inequalities. More specifically we prove

THEOREM 4.15. *Suppose that w is an $A_\infty(R^n)$ weight and that T is an operator which satisfies*

$$M^{\#}_{0,\alpha}\,\mathrm{Tf}(x) \leqslant c\,\mathrm{Mf}(x) \tag{4.16}$$

for α sufficiently small. Then for each f there is a polynomial q(Tf) (*of appropriate degree according to w, cf. Remark* 2.11) *such that*

$$\|\mathrm{Tf} - q(\mathrm{Tf})\|_{p,r;w} \leqslant c\,\|\mathrm{Mf}\|_{p,r;w}$$

for $1 < p \leqslant \infty$, $0 < r \leqslant \infty$ *or* $p = 1$ *and* $r = \infty$. *Also*

$$\|\mathrm{Tf}\|_{*} \leqslant c\,\|f\|_{\infty}.$$

Proof. That T is of type $(\infty, {}^*)$ is obvious. Moreover since

$$M^{\#}_{0,\alpha;w}\,\mathrm{Tf}(x) \leqslant c M^{\#}_{0,\alpha}\,\mathrm{Tf}(x)$$

we also have that

$$M^{\#}_{0,\alpha;w}\,\mathrm{Tf}(x) \leqslant c\,\mathrm{Mf}(x)$$

for α sufficiently small. Consequently by the weighted version of Corollary 2.6 it follows that there is q(Tf) such that

$$\|\mathrm{Tf} - q(\mathrm{Tf})\|_{p,r;w} \leqslant c\,\|M^{\#}_{0,\alpha;w}\,\mathrm{Tf}\|_{p,r;q} \leqslant c\,\|\mathrm{Mf}\|_{p,r;w}.$$

This completes the proof.

THEOREM 4.16. *Suppose that w is an $A_\infty(R^n)$ weight and that T is an operator which satisfies*

$$M^{\#}_{0,\alpha}\,\mathrm{Tf}(x) \leqslant c M^{\#} f(x) \tag{4.17}$$

for α sufficiently small. Then for each f there is a polynomial q(Tf) (*of appropriate degree according to w*) *such that*

$$\|\mathrm{Tf} - q(\mathrm{Tf})\|_{p,r;w} \leqslant c\,\|M^{\#} f\|_{p,r;w}$$

for $1 < p \leqslant \infty$, $0 < r \leqslant \infty$ *or* $p = 1$ *and* $r = \infty$. *Also*

$$\|\mathrm{Tf}\|_{*} \leqslant c\,\|M^{\#} f\|_{\infty}.$$

The proof, being identical to that of Theorem 4.15, is omitted.

Remark 4.17. Some instances of Theorems 4.15 and 4.16 are, of course, known. The reader may consult Muckenhoupt's survey paper [26] for further details. In this context, we only mention here that Miller [25] has discussed some weighted inequalities for $\psi.d.o$'s which are covered by Theorem 4.16. As for the weighted version of Aguilera's result the interested reader can verify that if w is a weight which satisfies the $A_p(R^n, k)$ condition for all k, i.e., if w satisfies the usual A_p condition over rectangles with ratio larger side–smaller side $= 2^k$, and if $c_k(p) =$ the $A_p(R^n, k)$ constant for w is such that $\sum_{k=1}^{\infty} 2^{-k} c_k(p) < \infty$, then for each f there is a polynomial $q(T^*_{\mathscr{F}} f)$ such that

$$\|T^*_{\mathscr{F}} f - q(T^*_{\mathscr{F}} f)\|_{p;w} \leqslant c \, \|f\|_{p;w}, \qquad 1 < p < \infty$$

and if $\sum_{k=1}^{\infty} 2^{-k} c_k(1)(\log 1/2^{-k} c_k(1)) < \infty$, then

$$\|T^*_{\mathscr{F}} f - q(T^*_{\mathscr{F}} f)\|_{1,\infty;w} \leqslant c \, \|f\|_{1;w}.$$

The proof of this remark is obvious and is therefore omitted. We warn the reader that we did not strive for the best possible result in this case.

Remark 4.18. It is possible to extend Theorems 4.6 and 4.7 to integral operators T which map $L^1_w(R^n)$ into $L^{1,\infty}_w(R^n)$ continuously. A straightforward condition implying that with $c = c(\alpha)$,

$$M^{\#}_{0,\alpha;w}\, \mathrm{Tf}(x) \leqslant c M_w f(x)$$

is given by

$$(A_{\Phi,w}) \qquad \sup_Q \frac{1}{w(Q)^2} \int_Q \int_Q |k(u, y) - k(v, y)| \; w(u)\, w(v)\, du\, dv \leqslant \Phi(y),$$

with Φ radial, nonincreasing, and in L^1_w (large values). Results of this nature have been discussed by Kurtz and Wheeden [24].

As for the weighted vector-valued estimates we have

THEOREM 4.19. *Suppose that w is an $A_\infty(R^n)$ weight and that $\{T_j\}_{j=1}^{\infty}$ is a sequence of operators verifying* (4.16) *uniformly, i.e., with a constant c independent of j. Then for functions $\{f_j\}$ there is a sequence $\{q_j(T_j f_j)\}$ of polynomials of appropriate degree, depending on w, such that*

$$\left\| \left(\sum_j |T_j f_j - q_j(T_j f_j)|^s \right)^{1/s} \right\|_{p,r;w} \leqslant c \left\| \left(\sum_j (\mathrm{Mf}_j)^s \right)^{1/s} \right\|_{p,r;w} \tag{4.18}$$

for $1 < p < \infty$, $1 \leqslant r < \infty$, $1 < s < \infty$, *and*

$$\left\| \left(\sum_j |T_j f_j - q_j(T_j f_j)|^s \right)^{1/s} \right\|_{1,\infty;w} \leqslant c \left\| \left(\sum_j (\mathrm{Mf}_j)^s \right)^{1/s} \right\|_{1,\infty;w} \tag{4.19}$$

Proof. The proof follows along the same lines as that of Theorem 4.15 using 2.13 and the weighted vector-valued version of the results of Fefferman and Stein and is therefore left for the interested reader.

Remark 4.19. Suppose that all the f_j's above save one, f_1 say, are 0. We can then use Muckenhoupt's results and estimate $\|\mathrm{Mf}_1\|_{1,\infty;w}$ by $\|f_1\|_{1;w}$ in (4.19) if (and only if) w is in $A_1(R^n)$ and use the result of Chung, Hunt, and Kurtz [10] and estimate $\|\mathrm{Mf}_1\|_{p,r;w}$ by $\|f_1\|_{p,r;w}$ in (4.19) if (and only if) w is in $A_{p,r}(R^n)=A_p(R^n)$. The same result is true in the vector-valued case. A simple way to see this is to observe that by Hölder's inequality $\mathrm{Mf}(x)\leqslant cM_{p;w}f(x)$, where

$$M_{p;w}f(x)=\sup_{x\in Q}\left(\frac{1}{w(Q)}\int_Q |f(y)|^p\, w(y)\, dy\right)^{1/p}$$

if w is in A_p, $1\leqslant p<\infty$. This immediately takes care of the case $p=1$ since the maximal theorem of Fefferman and Stein is true for doubling weights. Similarly for (4.19) we have only to recall that A_p implies $A_{p-\varepsilon}$, $p>1$ and some $\varepsilon>0$, and therefore $\mathrm{Mf}(x)\leqslant cM_{p-\varepsilon;w}f(x)$ as well. Another application of the maximal theorem of Fefferman and Stein establishes the desired conclusion in this case. These remarks provide a slight extension to the results of Anderson and John [2].

Remark 4.20. Although we do not pursue the matter here the techniques described above can be used to give vector-valued versions for sequences of operators $\{T_j\}$ verifying, uniformly, estimates such as (4.15) or (4.17), say. For instance in case (4.17) holds the right-hand side of (4.18) is replaced by $\|(\sum_j(M^{\#}f_j)^s)^{1/s}\|_{p,r;w}$ and that of (4.19) by $\|(\sum_j M^{\#}f_j)^s)^{1/s}\|_{j,\infty;w}$ respectively. Similarly for (4.15).

5. Spaces between L^∞ and BMO

We shall use this section to make some remarks pertaining to the couple $\bar{A}=(L^\infty, BMO)$:

Remark 5.1. Garnett and Jones [13] have computed the distance in BMO to L^∞. Recall that the John–Nirenberg inequality,

$$\sup_Q\left(\frac{1}{|Q|}\,|\{x\in Q\colon |f(x)-f_Q|>\lambda\}|\right)\leqslant e^{-\lambda/\varepsilon} \tag{5.1}$$

holds whenever f is in $BMO(R^n)$ and $\lambda>\lambda_0(\varepsilon, f)$. In fact, $\varepsilon=c\,\|f\|_*$ and

$\lambda_0 = c_1 \varepsilon$, with c and c_1 depending only on the dimension n, will do. Moreover, if $f \in L^\infty(R^n)$, then (5.1) obtains for all $\varepsilon > 0$. Setting

$$\varepsilon(f) = \inf\{\varepsilon > 0\text{: (5.1) holds}\},$$

Garnett and Jones showed that

$$\operatorname{dist}(f, L^\infty) = \lim_{t \to \infty} E(t, f; \bar{A}) \approx \varepsilon(f). \tag{5.2}$$

Now for a locally integrable function f and $0 < \alpha \leqslant 1$ set

$$\bar{M}^{\#}_{0,\alpha} f(x) = \sup_{x \in Q} \inf\{A \geqslant 0\colon |\{y \in Q\colon |f(y) - f_Q| > A\}| \leqslant \alpha\, |Q|\}.$$

With this notation, a careful application of a John–Nirenberg-type lemma and the argument of Garnett–Jones, shows that we have the following sharpening of (5.2):

$$K(t, f; \bar{A}) \approx \|\bar{M}^{\#}_{0,e^{-t}} f\|_\infty,$$

a result independently obtained by Svante Jansson.

This explicit evaluation of the K-function allows us to complete the statement of Rivière's interpolation theorem to include the case $p_1 = \infty$ as well. The reader can supply the needed details.

Remark 5.2. Bennett, DeVore, and Sharpley [3] have shown that the local Hardy maximal operator preserves *BMO*. An argument in the spirit of the proof of the basic inequality complemented by Theorem 3.4 shows that this is a consequence of the inequality

$$M^{\#}_{0,\alpha;Q_0}(\mathrm{Mf})(x) \leqslant c M_{0,\alpha;Q_0}(M^{\#} f)(x),$$

α sufficiently small.

Consequently the classes $\bar{A}_{\theta,r}$ are preserved by the local Hardy maximal operator as well.

Remark 5.3. Intermediate spaces between L^∞ and *BMO* arise naturally as the range of mappings such as the potential operator. The Riesz potential operator $I_\alpha f(x) = \int f(y)/|x-y|^{n\alpha}\, dy$, $0 < \alpha < 1$, maps $L^{p,1}(R^n)$ into $L^\infty(R^n)$ and $L^{p,\infty}(R^n)$ into $BMO(R^n)$, $p = 1/\alpha$, and by interpolation, $L^{p,p} = L^p(R^n)$ into $\bar{A}_{\theta,1/(1-\theta)}$ with $\theta = 1 - 1/p$.

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